Smooth Gauge Strings and $D \geq 2$ Lattice Yang-Mills Theories.

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Abstract

Employing the nonabelian duality transformation [25], I derive the Gauge String form of certain $D \geq 3$ lattice Yang-Mills (YM_D) theories in the strong coupling (*SC*) phase. With the judicious choice of the actions, in $D \geq 3$ our construction generalizes the Gross-Taylor stringy reformulation of the *continuous* YM_2 on a $2d$ manifold. Using the Eguchi-Kawai model as an example, we develope the algorithm to determine the weights $w[\tilde{M}]$ for connected *YM*-flux worldsheets $\tilde{M}$ immersed into the $2d$ skeleton of a $D \geq 3$ base-lattice. Owing to the invariance of $w[\tilde{M}]$ under a *continuous* group of area-preserving worldsheet homeomorphisms, the set of the weights $\{w[\tilde{M}]\}$ can be used to define the theory of the *smooth* *YM*-fluxes which unambiguously refers to a particular continuous YM_D system. I argue that the latter YM_D models (with a *finite* ultraviolet cut-off) for *sufficiently large* bare coupling constant(s) are reproduced, to all orders in $1/N$, by the smooth Gauge String thus associated. The asserted $YM_D/String$ duality allows to make a concrete prediction for the 'bare' string tension σ_0 which implies that (in the large N *SC* regime) the continuous YM_D systems exhibit confinement for $D \geq 2$. The resulting pattern is qualitatively consistent (in the extreme $D = 4$ *SC* limit) with the Witten's proposal motivated by the *AdS/CFT* correspondence.

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1 Introduction

The quest for a string representation of the $D = 4$ continuous Yang-Mills (YM_D) gauge theory shaped, to a large extent, many branches of the contemporary mathematical physics. Currently, there are two (qualitatively overlapping) candidates for the stringy systems conjecturaly dual to YM_D. The first one, due to Polyakov, puts forward certain Ansatz [1] for the world-sheet action which is to ensure the invariance of the Wilson loop averages $< W_C >$ with respect to the zig-zag *backtrackings* of the contour C. The complementary approach has recently sprung to life after daring conjecture of Maldacena [2] (further elaborated in [3, 4]) about the so-called AdF/CFT correspondence concerning the $\mathcal{N} = 4$ SUSY YM_4. Being motivated by the latter correspondence, Witten has made a speculative proposal [5] advocating that the ordinary nonsupersymmetric YM_4 exhibits confinement at least when considered for large N in the specific strong coupling (*SC*) regime. Namely, employing certain SUSY $YM_{\bar{D}}$, $\bar{D} > D$, properly broken to a nonsupersymmetric YM_D system, one is to fix an effective *finite* ultraviolet (*UV*) cut off Λ while keeping the relevant YM_D coupling constant(s) *sufficiently large*.

At present, the support of the conjectured duality-mappings is fairly limited to a few indirect though reasonably compelling arguments. On the other hand, final justification of that or another stringy representation calls either to reproduce the YM_D loop-equations in stringy terms or to provide with an explicit transformation of the gauge theory into a sort of string theory. It is our goal to make a step in the second direction approaching (with new tools) the old challenge: the exact reformulation of the continuous $D \geq 3$ YM_D theory in terms of the *microscopic* colour-electric *YM*-fluxes. More specifically, we focus on the pattern of the smooth Gauge String inherent in the presumable large N *SC* expansion (as *opposed* to the standard weak coupling (*WC*) series) valid in the *SC* regime similar to that of the Witten's proposal [5]. Complementary, the considered generic strongly coupled continuous YM_D models in $D = 4$ might be viewed as the prototypes of the effective low-energy YM_4 theory. The latter is supposed to result via the Wilsonian renormgroup (*RG*) flow of the *effective* actions (starting from the standard YM_4 in the *WC* phase) up to the confinement scale. In this perspective, the 'built in' *UV* cut off Λ in $D = 4$ is to be qualitatively identified

with a physical scale Λ_{YM_4} that is of order of the lowest glueball mass.

The short-cut way to a continuous model of smooth *YM*-fluxes is suggested by the two-dimensional analysis. Here, the pattern of the flux-theory [13] proposed by Gross and Taylor is encoded not only in the continuous YM_2 systems on a $2d$ manifold. It is also inherent in the associated RG invariant $2d$ *lattice* gauge models [20, 19] introduced via the plaquette-factor

$$Z(\{\tilde{b}_k\}|U) = \sum_R e^{-F(R)}\chi_R(U) \quad ; \quad e^{-F(R)} = dimR\; e^{-\Gamma(\{\tilde{b}_k\},N,\{C_p(R)\})}, \qquad (1.1)$$

where $dimR$, $\chi_R(U)$, and $C_p(R)$, $p = 1, ..., N$, stand respectively for the dimension, character and pth order Casimir operator associated to a given $SU(N)$ irreducible representation (irrep) R (while $\{\tilde{b}_k \sim N^0\}$ denotes a set of the dimensionless coupling constants). The key observation is that, in the $D \geq 3$ lattice YM_D systems (1.1), the pattern of the appropriately constructed flux-theory as well refers (owing to a subtle $D \geq 3$ 'descedant' of the $2d$ RG invariance) to the properly associated *continuous* YM_D models.

To support this assertion, I first reformulate the strongly coupled $D \geq 3$ lattice YM_D models (1.1) in terms of the Gauge String which does appropriately extend the $D = 2$ Gross-Taylor stringy pattern [13] into higher dimensions. This lattice theory of the *YM*-flux is endowed with certain *continuous* (rather than *discrete* as one might expect from the lattice formulation) group of the area-preserving homeomorphisms. In turn, the latter symmetry ensures that the considered $D \geq 2$ pattern of the lattice flux-theory can be employed to *unambiguously* define the associated $D \geq 2$ *smooth* Gauge String invariant under the area-preserving diffeomorphisms. The remarkable thing is that, in the latter continuous $D \geq 3$ flux-theory, one can identify (see Section 8) such *SC* conglomerates of the (piecewise) smooth flux-worldsheets which are in one-to-one correspondence with the judiciously associated varieties of the *WC* Feynman diagrams on the side of the properly specified continuous YM_D model. Therefore, the proposed smooth Gauge String provides with the concrete realization of the old expectation [9] (for a recent discussion see [5]) that certain nonperturbative effects 'close up' the windows of the Feynman diagrams trading the latter for the string worldsheets.

More specifically, building on the nonabelian duality transformation [25] recently proposed by the author, we show that in the $D \geq 2$ lattice systems (1.1) the free energy and the Wilson (multi)loop observables can be rewritten

in terms of the following statistics of strings. The lattice weight $w[\tilde{M}(T)]$ of a given connected worldsheet $\tilde{M}(T)$ (with the support on a subspace, represented by certain $2d$ cell-complex T, of the $2d$ skeleton of the D-dimensional base-lattice)

$$w[\tilde{M}(T)] = exp[-\bar{A}\Lambda^2\tilde{\sigma}_0(\{\tilde{b}_k\})\ ]\ N^{2-2h-b}\ J[\tilde{M}(T)|\{\tilde{b}_k\}]\ , \tag{1.2}$$

is composed of the three different blocks which altogether conspire so that the contribution of the strings with any *backtrackings* (i.e. foldings) is zero. In eq. (1.2), the first factor is the exponent of the Nambu-Goto term proportional to the total area $\bar{A}$ of $\tilde{M}$. The *D-independent* bare string tension $\sigma_0 = \Lambda^2\tilde{\sigma}_0$ (to be defined by eq. (1.6) below) is measured in the units of the *UV* cut off Λ squared (which, in $D \geq 3$, regularizes the 'transverse' string fluctuations). Next, there appears the standard 't Hooft factor, where h and b are respectively the genus and the number of the boundary contours $C_k,\ k = 1, ..., b,$ of $\tilde{M}$.

Finally, given a particular model (1.1), the third term $J[\tilde{M}(T)|\{\tilde{b}_k\}]$ (being equal to *unity* for a nonselfintersecting surface $\tilde{M}$) is sensitive only to the *topology* (but not to the geometry) of selfintersections of $\tilde{M}$. In particular, the dependence of $J[\tilde{M}(T)|\{\tilde{b}_k\}]$ on the coupling constants $\{\tilde{b}_k\}$ is collected from the elementary weights assigned to the admissible 'movable' singularities (to be specified later on) of the associated map $\varphi :\ \tilde{M} \rightarrow T$. As a result, the last term (similarly to the remaining ones) is invariant under the required continuous group of the area-preserving homeomorphisms so that the pattern (1.2) directly applicable to a *generic* smooth worldsheet $\tilde{M}$.

Next, consider the $D \geq 3$ continuous flux-theory defined as the statistics of the smooth worldsheets $\tilde{M}$ which are postulated to be endowed with the weights (1.2) corresponding to the smooth mappings of $\tilde{M}$ into the Euclidean space $\mathbf{R^D}$. In compliance with the above asserted *SC/ WC* correspondence, a given specification (1.1) the smooth Gauge String refers to the following *unique* continuous YM_D model. The *local* lagrangian of the latter is to be reconstructed as the D-dimensional 'pull-back'

$$L_2(F) \rightarrow L_D(F)\ :\ \{F^a_{\mu\nu}\ ;\ \mu,\nu = 1, 2\} \longrightarrow \{F^a_{\mu\nu}\ ;\ \mu,\nu = 1, ..., D\} \tag{1.3}$$

of the $D = 2$ lagrangian (composed of the $O(D)$-invariant combinations of

the $F^a_{\mu\nu} \equiv F$ tensor)

$$D = 2 \quad : \quad L_D(F) = \Lambda^{D-4} \sum_{n \geq 2} \sum_{r \in Y_n} \frac{[tr(F^k)]^{p_k}}{g_r(\{\tilde{b}_k\})} \ , \tag{1.4}$$

associated to such continuous YM_2 theory [19, 21] that its partition function on a $2d$ disc (of unit area with the free boundary conditions) is equal to the plaquette-factor (1.1). (In $D \geq 3$, within each trace of (1.4), the involved $(F_{\mu\nu})^i_j$-factors are prescribed to be totally symmetrized to exclude the $[F_{\mu\nu}, F_{\rho\sigma}]$-commutator dependent terms identically vanishing in $D = 2$.) To separate out the 'kinematical' *D-dependent* rescaling of the coupling constants, we have introduced the parameter Λ. The latter is to be identified with the effective *UV* cut off, for the considered strongly coupled YM_D systems, predetermined by the necessary regularization of the $D \geq 3$ *YM*-flux transverse fluctuations. Remark also that the *N*- and Λ-dependent coupling constants $g_r(\{\tilde{b}_k\}) \equiv g_r(\{\tilde{b}_k\}, N, \Lambda)$ are canonically labelled by the $S(n)$ irreps $r \in Y_n$ (parametrized by the partitions of n: $\sum_{k=1}^n k p_k = n$).

In sum, our proposal is that **the particular theory of the smooth Gauge String, thus induced through (1.1), reproduces (to all orders in $1/N$) the corresponding local continuous $D \geq 3$ YM_D model (1.4) for sufficiently large coupling constants** $\{g_r N^{2-\sum_{k=1}^n p_k} \sim O(N^0)\}$. (The latter constants are measured in the units of $1/N$ and Λ akin to eq. (9.1).) Note that in $D = 2$, after proper identification of the coupling constants, this correspondence is justified by the matching with the $2d$ pattern [13] of Gross and Taylor. One may also expect that the asserted $YM_D/String$ correspondence is unique to the extent that the stringy degrees of freedom are *directly* identified with the worldsheets of the *microscopic, conserved* *YM*-flux immersed (by mappings with certain admissible singularities to be specified after eq. (1.11) below) into a D-dimensional base-space.

The above constriant on $\{g_r\}$ (selecting the SC regime) accounts for the fact that in $D \geq 3$ the proposed $YM_D/String$ duality is limited by the stability of the *YM*-flux. In particular, the physical string tension σ_{ph} (entering e.g. the asymptotics of the Wilson loop averages) must be positive. As we will see, despite the extra $J[..]$-factor in eq. (1.2), the 'bare' and the physical string tensions (defined within the $1/N$ expansion) are conventionally

related so that the latter condition in the large N limit reads

$$\{ \ g_r(\{\tilde{b}_k\},..) \ \} \ : \quad \sigma_{ph} = (\tilde{\sigma}_0(\{\tilde{b}_k\}) - \zeta_D) \cdot \Lambda^2 \ > 0 \ , \tag{1.5}$$

i.e. the 'bare' part $\sigma_0 = \Lambda^2 \tilde{\sigma}_0$ should be larger than the $D \geq 3$ entropy contribution $\delta\sigma_{ent} = -\zeta_D \Lambda^2$ due to the transverse string fluctuations. Among our results, the central one is the explicit $\{\tilde{b}_k\}$-dependence of $\sigma_0 = \Lambda^2 \tilde{\sigma}_0(\{\tilde{b}_k\})$ entering the worldsheet weights (1.2). It is uniquely reconstructed

$$\Gamma(\{\tilde{b}_k\}, N, \{C_p(R)\}) = n\tilde{\sigma}_0(\{\tilde{b}_k\})(1 + O(N^{-1})) \quad ; \quad \sigma_0 = \Lambda^2 \tilde{\sigma}_0 \sim O(N^0), \tag{1.6}$$

from the formal $1/N$ asymptotics (with $R \in Y_n^{(N)}$, $n \sim O(N^0)$) of the admissible function $\Gamma(..)$ defining the associated lattice model (1.1). For example, the smooth Gauge String induced from the Heat-Kernal action [17]

$$Z(g^2|U) = \sum_{n \geq 0} \sum_{R \in Y_n^{(N)}} dimR \ \chi_R(U) \ exp[-\tilde{g}^2 C_2(R)/2] \ , \tag{1.7}$$

being presumably dual to the standard YM_D theory with $g^2 = \Lambda^{4-D} \tilde{g}^2 \sim 1/N$, refers to the following bare string tension (with $g^2 \sim 1/N$)

$$L_D(F) = tr(F_{\mu\nu}^2)/4g^2 \quad \Longleftrightarrow \quad \sigma_0 = g^2 N \Lambda^{D-2}/2 \ . \tag{1.8}$$

Summarizing, the predicted pattern (1.6) of σ_0 implies that the continuous $D \geq 3$ YM_D systems are confining at least in the large N *SC* regime belonging to the domain (1.5). Although in this regime the standard *WC* series are expected to fail, the latter *SC* phenomenon suggests a mechanism of confinement for YM_4 in the *WC* phase (see Conclusions). Remark also that the physical string tension in eq. (1.5) generically is *not* adjusted to be infinitely less than the squared *UV* cut off Λ (which in $D = 4$ matches with the above identification $\Lambda \sim \Lambda_{YM_4}$). Actually, in the considered YM_D models the introduced smooth flux-worlsheets might become unstable degrees of freedom even *prior* to the saturation of (1.5) owing to the presumable large N phase transition(s) (generalizing the $D = 2$ situation [12]).

Finally, the distinguished regime of the smooth Gauge String is the extreme large N *SC* limit where $\sigma_0(\{\tilde{b}_k\}) >> \Lambda^2$. As in eq. (1.5) the entropy-constant ζ_D is $\{\tilde{b}_k\}$-independent, in this limit the leading order of the Wilson loop averages $< W_C >$ is given by the minimal area contiribution that

allows for a number of nontrivial predictions. First of all, the physical string tension in this regime merges with the 'bare' one, $\sigma_{ph} \to \sigma_0$, determined by eq. (1.6). (We will show in Section 9 that the Witten's SC asymptotics [5] of σ_{ph} in $D=4$ is semiquantitatively consistent with the latter prediction). Also, provided the minimal area worldsheet $\tilde{M}_{min}(C)$ has the support $T_{min}(C)$ on a $2d$ manifold (rather than on a $2d$ cell-complex), in this limit the pattern of the $D \geq 3$ averages $< W_C >$ is reduced to the one in the corresponding continuous $D=2$ YM_2 theory (1.4) conventionally defined on $T_{min}(C)$.

1.1 The $D \geq 2$ pattern of the Gauge String weights.

A. The D = 2 case.

Recall that, in the $2d$ framework initiated by Gross and Taylor, the partition function $\tilde{X}_M$ of a given continuous YM_2 theory on a $2d$ manifold M of the area A (without boundaries) is rewritten as the sum

$$\tilde{X}_M = \int_{\tilde{M}} df \; (-1)^{P_f} \, \frac{N^{\chi_f}}{|C_f|} \, exp[-\sigma_0 n_f A] \quad ; \qquad f: \; \tilde{M} \to M, \tag{1.9}$$

over the topologically distinct branched covering spaces $\tilde{M}$ of M specified by the mappings *f*. The latter maps locally satisfy the definition [26, 23] of the immersion $\tilde{M} \to M$ everywhere on M except for a set of isolated points where the singularities, corresponding to a branch-point or/and to a collapsed (to a point) subsurface connecting a few sheets, are allowed. To visualize the pattern of $\tilde{M}$ as the associated Riemann surface (*without* foldings but with the singularities to be viewed as certain collapsed $2d$ subsurfaces) identified with a particular string worldsheet $\tilde{M}$ of the total Euler characteristic χ_f, I propose the cutting-gluing rules which readily generalize to the $D \geq 3$ case. As for $|C_f|, P_f$, and n_f, in eq. (1.9) they denote respectively the symmetry factor (i.e. the number of distinct automorphisms κ of f : $f \circ \kappa = f$), the 'parity', and the degree (i.e. the number of the covering sheets) of the map f. The sum (1.9) includes all admissible disconnected contributions in such a way that the free energy $-ln[\tilde{X}_M]$ is supposed to be deduced [13] from (1.9) constraining the worldsheets $\tilde{M}$ to be connected.

To make contact with the $T = M,\ \bar{A} = n_f A$ case of (1.2), observe first that the summation (1.9) over the maps f effectively enumerates the admissible selfintersections of $\tilde{M}$. In particular, the sum over the singularities of f implies the one over the 'nonmovable' and (the positions of) the 'movable' branch points parametrized by the corresponding cyclic decomposition $\{p\}$: $\sum_{k=1}^{n} kp_k = n_f$ of n_f. These two types of the points are respectively assigned with the nontrivial weights $\tilde{w}_{\{p\}}$ and $w_{\{p\}}(\{\tilde{b}_k\})$ (included into the measure df of (0.5)) which are $\{\tilde{b}_k\}$-dependent only in the 'movable' case. (The $w_{\{p\}}$-, $\tilde{w}_{\{p\}}$-factors, together with $(-1)^{P_f}/|C_f|$, are collected into $J[\tilde{M}(T)|\{\tilde{b}_k\}]$ of (1.2).) Therefore, the summation df implies in particular the multiple integrals

$$\prod_{\{p\}} \frac{[w_{\{p\}}(\{\tilde{b}_k\})]^{m_{\{p\}}}}{[m_{\{p\}}]!} \prod_{k=1}^{m_{\{p\}}} \int_M d^2 X_{\{p\}}^{(k)} \tag{1.10}$$

over all positions $X_{\{p\}}^{(k)}$ on M of (a given number $m_{\{p\}}$ of) the movable branch points that introduces additional dependence on the area $A = \int_M d^2 X_{\{p\}}^{(k)}$ of M. ('Nonmovable' singularities of the f-map can be placed anywhere on M but do not carry any area- or $\{\tilde{b}_k\}$-dependent factors.) Finally, the bare string tension $\sigma_0 = \Lambda^2 \tilde{\sigma}_0(\{\tilde{b}_k\})$ (defined via eq. (1.6)) enters the exponent of (1.9) as the *no-fold variant* of the Nambu-Goto term. Our attention is mostly confined to the option (1.8) corresponding to the $D \geq 2$ $SU(N)$ Heat-Kernal lattice gauge theory (1.7). In this case, $\sigma_0 = \Lambda^2 \tilde{g}^2 N/2$ and there are only the simple transposition branch points (i.e. the ones connecting a pair of the sheets) weighted by $w_2 = \tilde{g}^2 N$.

Finally, the $D = 2$ representation (1.9) of the continuous YM_2 theories (1.4) is valid also in the corresponding $2d$ *lattice* gauge systems (1.1) on a discretized surface M. The origin of this two-fold interpretation of (1.9) resides in the RG invariance of the latter lattice models. Complementary, owing to the symmetry of the continuous YM_2 models (1.4) under the area-preserving diffeomorphisms, the RG invariance of (1.1) results in the invariance of the lattice weights $w[\tilde{M}]$ under the *continuous* group of the area-preserving worldsheet homeomorphisms. These homeomorphisms *continuously* (rather than discretely) translate the positions of the singularities of the map f everywhere on M including *interiors* of the plaquettes.

B. Extension to the D $\geq$ 3 case.

A given $D \geq 3$ lattice YM_D system (1.1) (having some lattice $\mathbf{L^D}$ as the base-space B) can be equally viewed as the YM model (1.1) defined on the $2d$ skeleton $\mathbf{T^D}$ of $\mathbf{L^D}$ represented by the associated $2d$ cell-complex. As we will discuss in Section 7, this reformulation allows to reveal certain $D \geq 3$ 'descedant' of the $2d$ RG invariance of (1.1) that in turn foreshadows the existence of the $D \geq 3$ extension of the pattern (1.9).

Consider the partition function $\tilde{X}_{L^D}$ of the YM_D system (1.1) defined on $B = \mathbf{L^D}$. Combining the nonabelian duality transformation [25] with the methods of algebraic topology [26], we derive that the pattern (1.9) remains to be valid in $D \geq 3$ with the only modification. It concerns the structure of the relevant mappings (to be summed over) specifying the admissible topology of the worldsheets $\tilde{M}$ of the *YM*-flux. The $D = 2$ maps f are superseded by the mappings φ

$$\varphi : \quad \tilde{M} \longrightarrow T \quad \in \quad \mathbf{T^D}, \tag{1.11}$$

of a $2d$ surface $\tilde{M}$ onto a given subspace T (represented by a $2d$ cell-complex) of the $2d$ skeleton $\mathbf{T^D}$ of the D-dimensional base-lattice $\mathbf{L^D}$. Akin to the $D = 2$ case, the maps φ locally comply with the definition [26, 23] of the immersion $\tilde{M} \to T$ anywhere except for a countable set of points on T where certain singularities are allowed. The relevant (for the smooth implementation of the Gauge String) ones are either of the *same* type as in the $2d$ f-mappings (1.9), i.e. the branch points or/and the collapsed subsurfaces, or of the type corresponding to the *homotopy retraction* (see e.g. Appendix C) of the latter irregularities. In particular, the included into $d\varphi$-measure $\{\tilde{b}_k\}$-dependent weights $w_{\{p\}}(\{\tilde{b}_k\})$ (of the movable branch points) in $D \geq 3$ are equal to their counterparts in the $2d$ case (1.10) associated to the same continuous YM_2 theory (1.4). (Akin to eq. (1.9), one can argue that the free energy $-ln[\tilde{X}_{L^D}]$ is provided by the restriciton of the sum for partition function $\tilde{X}_{L^D}$ to the one over the connected worldsheets $\tilde{M}$.)

The $D \geq 3$ nature of the mappings (1.11) is reflected by the fact that, owing to possible 'higher-dimensional' selfintersections of $\tilde{M}(T)$, the taget-space T generically is represented by the $2d$ cell-complex $T = \cup_k E_k$ rather than by a $2d$ surface. (Recall that any $2d$ cell-complex, after cutting out along the links shared by *more* than two plaquettes, becomes [17, 26] a disjoint union of $2d$ surfaces E_k of the areas A_k with certain boundaries. Conversely, the set $\{E_k\}$ can be combined back into $T = \cup_k E_k$ according

to the incidence numbers [17] corresponding to the entries of the associated *incidence-matrix.*) Therefore, the construction of $\tilde{M}(T)$ (via our cutting-gluing algorithm developed for the canonical branched coverings (1.9)) entails the appropriate generalization of the notion of the Riemann surface (with the total area $\bar{A} = \sum_k n_\varphi^{(k)} A_k$). Alternatively, $\tilde{M}(T)$ can be viewed as the specific *generalization* of the branched covering space (wrapped around T) suitable for the analysis of the Gauge String.

Next, as it is shown in Section 8, the total contribution of the worldsheets $\tilde{M}$ with arbitrary *backtrackings* (that, generalizing the $D = 2$ case, bound any zero 3-volume) *vanishes.* The simplest $J[\tilde{M}(T)|\{\tilde{b}_k\}] = 1$ pattern (1.2) of $w[\tilde{M}]$ in $D \geq 2$ arises when the genus h connected string worldsheet $\tilde{M}(T)$ is represented by an *embedding* $T = \tilde{M}(T)$ (i.e. $\tilde{M}(T)$ does *not* selfintersect). Nevertheless, the set of all possible $J[\tilde{M}(T)|\{\tilde{b}_k\}] = 1$ embedding-weights (1.2) does *not* discriminate between those distinct models (1.1)/(1.4) which provide with one and the same bare string tension (1.6). To distinguish between the different models, *the specification of the remaining immersion-weights (assigned to the selfintersecting string worlsheets) is indispensable.* In this way, one reconstructs the data encoded in a given continuous YM_2 theory (1.4) that originally served as the bridge (1.3) between the Gauge String and the corresponding continuous YM_D model.

Owing to the asserted pattern, the $D \geq 3$ lattice weights $w[\tilde{M}(T)]$ are invariant under certain continuous group of the area-preserving homeomorphisms extending the $D = 2$ ones. As a result, the set $\{w[\tilde{M}(T)]\}$ can be unambiguously used to introduce the statistics of the (piecewise) *smooth YM* flux-worldsheets $\tilde{M}(T)$, with the latter homeomorphisms being traded for the corresponding diffeomorphisms. As for the sum over the worldsheets, it is specified by the one running over the φ-mappings (1.11). This time, one is to consider the piecewise smooth immersions (with the admissible singularities which, in $D \geq 4$, can be restricted to the ones listed after eq. (1.11)) of the $2d$ *manifolds* M into the D-dimensional space-time $B = \mathbf{R^D}$ that results in the worldsheet $\tilde{M}$ with the support on $T \in \mathbf{R^D}$. In turn, it constitutes the proper class of the $2d$ cell-complexes T which can play the role of the taget-spaces for the considered (piecewise) *smooth* maps φ.

As we will discuss in Section 8, the $D \geq 3$ dynamics of the smooth Gauge String in certain sense is dramatically *simpler* compared to its lattice counterpart. First of all, it is convenient to make certain redefinition of the bare

string tension (that amounts to the substitution $\tilde{\sigma}_0 = \lambda/2 \to \lambda(1-1/N^2)/2$ in the case of (1.7)) to get rid of a redundant subset of the 'movable' singularities present in the $SU(N)$ mappings (1.11) even for a 1-sheet covering of T. Given this modification, consider the subset of the smooth worldsheets $\tilde{M}(T)$ (without boundaries) which are constrained to be strictly nonselfintersecting in $D \geq 5$ and allowed to selfintersect at an arbitrary union of isolated points in $D = 4$. (Actually the latter condition can weakened e.g. to include non(self)intersecting boundary contour(s)). They are assigned with the simplest $J[\tilde{M}(T)|\{\tilde{b}_k\}] = 1$ weight-pattern (1.2). The point is that **in $D \geq 4$ the latter subset is dense in the set of all worldsheets $\tilde{M}$ parametrized by the (piecewise) smooth mappings (1.11) into $\mathbf{R^D}$**, provided the latter redefinition of $\tilde{\sigma}_0$. In particular, it justifies (at least in $D \geq 4$) the validity of the simple relation (1.5). As for the more complicated pattern (1.2) of the weights, in $D \geq 4$ it is observable for example within such Wilson loop averages $< W_C >$ where the corresponding minimal surface $S_{min}(C)$ selfintersects (or when the boundary contour C has some zig-zag backtrackings).

2 Outline of the further content.

To make the analysis of the $D \geq 2$ lattice Gauge String more concise, we employ the Twisted Eguchi-Kawai (TEK) representation [10] of the large N 'infinite-lattice' *SU(N)* gauge systems. Recall that in the limit $N \to \infty$ the partition function (PF) $\tilde{X}_{L^D}$ of a lattice YM_D theory like (1.1) in a $D = 2p$ volume $L^D = N^2$ can be reproduced (at least within both the *SC* and the *WC* series)

$$\lim_{N\to\infty} \tilde{X}_{L^D} = \lim_{N\to\infty} \left(\int \prod_{\rho=1}^{D} dU_\rho \prod_{\mu\nu=1}^{D(D-1)/2} Z(g^2|t \cdot U_\mu U_\nu U_\mu^+ U_\nu^+) \right)^{L^D} , \tag{2.1}$$

through the PF $\tilde{X}_D$ of the associated D-matrix *SU(N)* TEK model with the reduced space-time dependence (where $t = \exp[i2\pi/N^{\frac{2}{D}}] \in \mathbf{Z_N}$). Therefore, in the $N \to \infty$ *SC* phase, the free energy of (2.1) yields the generating functional for the YM_D string-weights assigned to the worldsheets corresponding to the $2d$ spheres immersed into $\mathbf{L^D}$. On the side of the TEK model, the

surfaces are 'wrapped around' the EK base-lattice T_{EK} which has the topology of the $2d$-*skeleton* of the D-dimensional cube with periodic boundary conditions. Thus, T_{EK} is homeomorphic to the $2d$ cell-complex visualized as the union

$$T_{EK} = \cup_{\mu\nu=1}^{D(D-1)/2} E_{\mu\nu} \quad , \quad E_{\rho\mu} \cap E_{\rho\nu} = l_\rho \ , \tag{2.2}$$

of the $D(D-1)/2$ mutually intertwined 2-tora $E_{\mu\nu}$ sharing D uncontractible cycles (i.e. compactified links of the D-cube) l_ρ in common.

To proceed further, let us first label the *SU(N)* irreps summed up in each $\mu\nu$-species of the *Z*-factor (1.1) entering (2.1) by

$$R_{\mu\nu} \in Y^{(N)}_{n_{\mu\nu}} \quad , \quad n_+ = \sum_{\mu\nu=1}^{D(D-1)/2} n_{\mu\nu}. \tag{2.3}$$

It is convenient to rewrite the *SU(N)* TEK PF (2.1) in the following form

$$\tilde{X}_D = \sum_{\{n_{\mu\nu}\}} \sum_{\{R_{\mu\nu} \in Y^{(N)}_{n_{\mu\nu}}\}} t^{n_+} e^{-S(\{R_{\mu\nu}\})} B(\{R_{\mu\nu}\}) = \sum_{\{n_{\mu\nu}\}} t^{n_+} B(\{n_{\mu\nu}\}), \tag{2.4}$$

introducing the elementary master-integrals $B(\{R_{\mu\nu}\})$ (which are then composed into $B(\{n_{\mu\nu}\})$)

$$B(\{R_{\mu\nu}\}) = \int \prod_{\{\rho\}} dU_\rho \prod_{\{\mu\nu\}} \chi_{R_{\mu\nu}}(U_{\mu\nu}) \ \ ; \ \ e^{-S(\{R_{\mu\nu}\})} = \prod_{\{\mu\nu\}} e^{-F(R_{\mu\nu})}, \tag{2.5}$$

where $U_{\mu\nu} \equiv U_\mu U_\nu U^+_\mu U^+_\nu$, $F(R)$ is defined by eq. (1.1), and we have used the identity $\chi_R(tV) = t^{n(R)}\chi_R(V)$, $t \in Z_N$, $R \in Y^{(N)}_{n(R)}$. In the *SC* phase, we are concerned below, the twist-factor t in eq. (2.4) is irrelevant [10] in the limit $N \to \infty$ so that the TEK model is reduced to the original Eguchi-Kawai one. In this regime, the $t = 1$ correspondence (2.1) is supposed to be valid in any (not necessarily even) $D \geq 2$.

Next, making use of the nonabelian duality transformation [25], in Section 3 we rewrite the TEK master-integral $B(\{n_{\mu\nu}\})$ as the weighted sum of the Tr_{n_+}-characters (i.e. traces)

$$B(\{n_{\mu\nu}\}) = \sum_{\{R_{\mu\nu} \in Y^{(N)}_{n_{\mu\nu}}\}} e^{-S(\{R_{\mu\nu}\})} \ Tr_{n_+}[\mathbf{D}(\ A_{n_+}(\{R_{\mu\nu}\}) \ )] \ , \tag{2.6}$$

of certain master-elements $A_{n_+}(\{R_{\mu\nu}\}) = \sum_{\sigma\in S(n_+)} a(\sigma|\{R_{\mu\nu}\})\ \sigma$. Being defined by eqs. (3.34),(3.35), they take values in the *tensor* representation of the $S(n_+)$ algebra (with n_+ being given by eq. (2.3)). The latter is deduced by linearity from the canonical representation [16] for $S(n)$-group elements σ

$$\mathbf{D}(\sigma)^{\{i^{\oplus n}\}}_{\{j^{\oplus n}\}} = \delta^{i_{\sigma(1)}}_{j_1}\delta^{i_{\sigma(2)}}_{j_2}...\delta^{i_{\sigma(n)}}_{j_n} \quad ; \quad \hat{\sigma}:\ k\to\sigma(k)\ ,\ k=1,...,n, \tag{2.7}$$

where δ^i_j denotes the 'N-dimensional' Kronecker delta function.

To relate (2.6) with the stringy pattern like (1.9), in Section 4 we represent this equation in the form suitable for the algebraic definition of the topological data. First of all, one observes (see Appendix B) that the Tr_{n_+}-trace in eq. (2.6) is simply related to the associated character of the *regular* $S(n_+)$-representation. As a result, $B(\{n_{\mu\nu}\})$ can be rewritten as the delta-function on the $S(n_+)$-algebra (that selects the contribution of the weight of the $S(n_+)$ unity-permutation $\hat{1}_{[n_+]}$)

$$B(\{n_{\mu\nu}\}) = Tr_{n_+}[\mathbf{D}(\tilde{A}_{n_+})] = \delta_{n_+}(\ \Lambda^{(1)}_{n_+}\ \tilde{A}_{n_+}\ ) \tag{2.8}$$

that already played the important role in the $D=2$ analysis [13]. Next, employing the Schur-Weyl duality, *both* the operator $\Lambda^{(1)}_{n_+}$ (defined by the *U(N)* variant of eq. (3.9), see Section 3) *and*

$$\tilde{A}_{n_+} = \sum_{\{R_{\mu\nu}\in Y^{(N)}_{n_{\mu\nu}}\}} e^{-S(\{R_{\mu\nu}\})}\ A_{n_+}(\{R_{\mu\nu}\}) \tag{2.9}$$

are reformulated entirely in terms of the symmetric group elements (i.e. no *SU(N)* irreps R_ϕ are left). The resulting expression (4.8) is suitable to specify the mappings (1.11). The $D\geq 3$ nature of (1.11) is reflected by the *outer*-product structure of $\tilde{A}_{n_+}$ which is represented as certain combination of the $S(n_\phi)$-blocks embedded to act in the common *enveloping* space of the $S(n_+) = \cup_\phi S(n_\phi)$ algebra (with $\phi\in\{\mu\nu\},\{\rho\}$). The technique, dealing with such compositions, is naturally inherited from the nonabelian duality transformation [25].

The remaining material is organized as following. In Section 5, we rederive by our methods the Baez-Taylor reformulation [27] of the Gross-Taylor $2d$ pattern (1.9) for the Heat-Kernal model (1.7). We also briefly sketch how

our method generalizes for any admissible model (1.1). Being more compact than the Gross-Taylor one, the $D = 2$ representation of the Baez-Taylor type has the structure where the full stringy pattern is not entirely manifest. To circumvent this problem we formulate a simple prescription how to transform the latter into the former.

The $D \geq 3$ generalization of the $2d$ Baez-Taylor representation [27] is derived in Section 6 for the Eguchi-Kawai models (2.1) which fully justifies the announced pattern (1.11) of the $D \geq 3$ mappings. (In particular, we make certain conjecture concerning the concise topological reinterpretation, generalizing the $D = 2$ one [18], of the $D \geq 3$ sums like (1.9) in the formal topological limit when $\Gamma(..) \rightarrow 0$.) As well as in the $D = 2$ case, to relate the obtained $D \geq 3$ representation with the manifest stringy pattern (1.2), a natural extension of the $D = 2$ prescription (formulated in Section 5) is suggested. In Section 7, we discuss the structure of the asserted *continuous* group of the area-preserving homeomorphisms (the generic lattice Gauge String weights $w[\tilde{M}]$ are endowed with) and make a brief comparison with the earlier large N variants [8, 7] of the Wilson's *SC* expansion [6] devoid of the latter invariance. In Section 8, we discuss the major qualitative features of the Gauge String accentuating the similarities and differences between the continuous flux-theory and the conventional paradigm of the $D \geq 3$ 'fundamental' strings. In the last section, we put forward a speculative proposal for the mechanism of confinement in the standard weakly-coupled continuous gauge theory (1.8) at large N. Also a preliminary contact with the two existing stringy proposals [5, 1] is made. Finally, the Appendices contain technical pieces of some derivations used in the main text.

3 The Dual Representation of $\tilde{X}_D$.

To derive (2.6), we apply the nonabelian duality transformation to the partition function of the *SU(N)* TEK model (2.1)/(1.1) following the general algorithm formulated in [25] for a generic *SU(N)* D-matrix system. To begin with, the $SU(N)$ character is to be represented in the form (see e.g. [18, 25]) reminiscent of the one of eq. (2.6). Let C_R denote the canonical

Young idempotent [16] proportional, $P_R = d_R C_R$, to the Young projector

$$P_R = \frac{d_R}{n!} \sum_{\sigma \in S(n)} \chi_R(\sigma)\ \sigma \qquad , \qquad R \in Y_n \ , \tag{3.1}$$

where $\chi_R(\sigma)$, d_R are the character and the dimension associated to the $S(n)$ irrep R (while $[P_R, \sigma] = 0$, $\forall \sigma \in S(n)$). Then, $\chi_R(U)$ assumes (akin to (2.6)) the form of the trace: $\chi_R(U) = Tr_n[\mathbf{D}(C_R)U^{\oplus n}]$, $R \in Y_n^{(N)}$, where

$$Tr_n[\mathbf{D}(\sigma)U^{\oplus n}] = \sum_{i_1 i_2 .. i_n = 1}^{N} U_{i_1}^{i_{\sigma(1)}} U_{i_2}^{i_{\sigma(2)}} ... U_{i_n}^{i_{\sigma(n)}} \ , \tag{3.2}$$

and the tensor $\mathbf{D}(\sigma)$ is defined in eq. (2.7).

Altogether, it implies that the master-integral (2.5) can be rewritten in the synthetic form

$$B(\{R_{\mu\nu}\}) = \int Tr_{4n_+}[\mathbf{D}(\Xi_{4n_+}(\{R_{\mu\nu}\}))\ \mathbf{D}(\{U_\rho \otimes U_\rho^+\})] \prod_{\tilde{\rho}=1}^{D} dU_{\tilde{\rho}} \tag{3.3}$$

where the $S(4n_+)$-algebra valued tensor $\mathbf{D}(\Xi_{4n_+}(\{R_{\mu\nu}\}))$ (to be defined by eq. (3.19) below) multiplies the complementary tensor given by the ordered direct product of the $4n_+$ elementary $N \times N$ matrices $(U_\rho)^i_j$, $(U_\rho^+)^k_l$:

$$\mathbf{D}(\{U_\rho \otimes U_\rho^+\}) \equiv \bigotimes_{\rho=1}^{D} \Big(\ (U_\rho)^{\oplus n_\rho} \otimes (U_\rho^+)^{\oplus n_\rho} \Big) \qquad , \qquad 2n_+ = \sum_{\rho=1}^{D} n_\rho \ . \tag{3.4}$$

Given $\Xi_{4n_+}(\{R_{\mu\nu}\})$, we first derive the intermediate $S(4n_+)$ representation

$$B(\{R_{\mu\nu}\}) = Tr_{4n_+}[\ \mathbf{D}(\ J_{4n_+}(\{R_{\mu\nu}\})\ )\ ] \tag{3.5}$$

which in Section 3.3 will be transformed into the final $S(n_+)$ form of eq. (2.6). To determine the operator $J_{4n_+} \in S(4n_+)$, in (3.3) one is to substitute the dual form [25] of the *SU(N)* measure, i.e. to represent the result of the D different U_ρ-integrations as an $S(4n_+)$-tensor akin to $\mathbf{D}(\Xi_{4n_+}(\{R_{\mu\nu}\}))$.

Before we explain the latter procedure, let us make the pattern of the element $\Xi_{4n_+}(\{R_{\mu\nu}\}) = \sum_{\sigma \in S(4n_+)} \xi(\sigma|\{R_{\mu\nu}\})\ \sigma$ more transparent relating the explicit form of the $S(4n_+)$-group tensor $\mathbf{D}(\sigma)$ with the one of (3.4).

To this aim, we introduce first a particular $S(4n_+)$ permutation $\alpha_{\{n_\rho\}}$ via the mapping $m \to \alpha(m)$, $m = 1, ..., 4n_+$. Then, generalizing the tensor representation (2.7), the tensor $\mathbf{D}(\alpha_{\{n_\rho\}})$ stands for

$$\delta^{i_{\alpha(1)}}_{j_1} ... \delta^{i_{\alpha(n_1)}}_{j_{n_1}} \delta^{k_{\alpha(n_1+1)}}_{l_{n_1+1}} ... \delta^{k_{\alpha(2n_1)}}_{l_{2n_1}} ... \delta^{k_{\alpha(4n_+-2n_D)}}_{l_{4n_+-2n_D}} ... \delta^{k_{\alpha(4n_+-n_D)}}_{l_{4n_+-n_D}} ... \delta^{k_{\alpha(4n_+)}}_{l_{4n_+}} \ . \tag{3.6}$$

where, to each individual U_ρ-, U^+_ρ-factor in the product (3.4), we associate one copy of the Kronecker delta-function. In this way, both $\mathbf{D}(\alpha_{\{n_\rho\}})$ and the block (3.4) are defined to act on one and the same $S(4n_+)$ space (to be manifestly constructed, see eqs. (3.13),(3.14) below). Complementary, the pattern of the trace (3.5) of $J_{4n_+}(\{R_{\mu\nu}\})$ (defined through the structures like (3.6)) naturally generalizes the above convention (3.2) to the case of (3.4).

3.1 The Dual form of the $SU(N)$ measure.

On a given base-lattice, a generic multilink integral (like in (2.1)) evidently can be expressed [17] in terms of the 1-link integrals $M^G(n,m)^{p_1...q_m}_{j_1...l_m}$:

$$\int (U)^{p_1}_{j_1} ... (U)^{p_n}_{j_n} \ (U^+)^{q_1}_{l_1} ... (U^+)^{q_m}_{l_m} dU \equiv \int \mathbf{D}(U)^{\{p^{\oplus n}\}}_{\{j^{\oplus n}\}} \ \mathbf{D}(U^+)^{\{q^{\oplus m}\}}_{\{l^{\oplus m}\}} dU \tag{3.7}$$

composed of the $N \times N$ matrices $(U)^{p_k}_{j_k}$, $(U^+)^{q_k}_{l_k}$ in the (anti)fundamental representation of the considered Lie group G. As we will show in a moment, the $SU(N)$ TEK partition function (PF) (2.1) is invariant under the substitution of the $SU(N)$ link-variables by the $U(N) = [SU(N) \otimes U(1)]/\mathbf{Z_N}$ ones. In the *U(N)* case, where the dual [25] form of $M^{U(N)}(n,m)$ in terms of the $S(n)$-valued tensors (2.7) reads

$$M^{U(N)}(n,m)^{p_1...q_m}_{j_1...l_m} = \delta[n,m] \sum_{\sigma \in S(n)} \mathbf{D}(\sigma^{-1}\Lambda^{(-1)}_n)^{\{q^{\oplus n}\}}_{\{j^{\oplus n}\}} \ \mathbf{D}(\sigma)^{\{p^{\oplus n}\}}_{\{l^{\oplus n}\}} \ . \tag{3.8}$$

that renders manifest the previously known interrelation [17, 13] between (3.7) and the symmetric group's structures. The operator $\Lambda^{(-1)}_n \in S(n)$ belongs to the (*U(N)* option of the) family [25]

$$\Lambda^{(m)}_n = \sum_{R \in Y^{(N)}_n} d_R \ (n! \ dimR/d_R)^m \ C_R \quad , \quad m \in \mathbf{Z} \ , \tag{3.9}$$

where $C_R = P_R/d_R$ is defined by eq. (3.1). In eq. (3.9), d_R and $dimR$ are respectively the dimension of the $S(n)$-irrep and chiral $U(N)$-irrep both described by the same Young tableau $Y_n^{(N)}$ containing not more than N rows. (In eqs. (4.4),(4.8) below, we will consider the *SU(N)* variant of $\Lambda_n^{(m)}$ where the sum is traded for the one over the *SU(N)* irreps.)

As for the possibility to substitute the $SU(N)$ TEK link-variables by the $U(N) = [SU(N) \otimes U(1)]/\mathbf{Z_N}$ ones, the pattern of eq. (2.1) ensures that the TEK action is invariant under the D copies of the *extended* transformations $[U(1)]^{\oplus D}$: $U_\rho \rightarrow t_\rho U_\rho$, where $t_\rho \in U(1)$ rather than taking value in the center-subgroup $\mathbf{Z_N}$ of $SU(N)$. As a result, the nondiagonal moments $M^{SU(N)}(n,m)$, $n \neq m$, do *not* contribute into the TEK PF $\tilde{X}_D$. As for the remaining diagonal integrals $M^{SU(N)}(n,n)$, the latter extended invariance justifies [25] the required substitution: $M^{SU(N)}(n,n) = M^{U(N)}(n,n)$, $\forall n \in \mathbf{Z}_{\geq 0}$. Moreover, in the context of the large N *SC* expansion, in eq. (3.9) one can insert the *SU(N)* variant of $\Lambda_n^{(-1)}$. (The difference can be traced back to the contributions of the *SU(N)* string-junctions which are supposed to be irrelevant to all orders in *1/N*.)

Actually, the derivation of (3.5) calls for the alternative $S(2n)$ reformulation [25] of the $S(n) \otimes S(n)$ formula (3.8) that will require the explicit form of an $S(2n)$-basis. For this purpose, we first recall that each individual matrix U_ρ^+ or U_ρ can be viewed [16] as the operator acting on the associated elementary *N*-dimensional subspace $|i_\pm(\rho)>$ according to the pattern: $\hat{U}|i_- >= \sum_{j_-=1}^N U_{i_-}^{j_-}|j_- >$ and similarly for $\hat{U}^+|i_+ >$. Complementary, for a given $\sigma \in S(n)$, the operator (2.7) acts as the corresponding permutation of the elementary subspaces $|i_k >$

$$\hat{\sigma}|i_1>|i_2>...|i_n>=|i_{\sigma^{-1}(1)}>|i_{\sigma^{-1}(2)}>...|i_{\sigma^{-1}(n)}>\equiv \mathbf{D}(\sigma)^{\{j^{\oplus n}\}}_{\{i^{\oplus n}\}}|j>^{\oplus n} \tag{3.10}$$

where in the r.h.s. the summation $\sum_{\{j_k\}}$ is implied. As a given $S(4n_+)$-basis is constructed as the outer product of the $4n_+$ building blocks $|i_\pm(\rho)>$ (*ordered* according to a particular prescription), the elementary subspaces $|i_k>$ of eq. (3.10) are represented by $|i_\pm(\rho)>$.

Returning to the $S(2n)$-reformulation of eq. (3.8), in the basis $|I_{2n}>= |I_n^{(+)}> \otimes |I_n^{(-)}>$ (with $|I_n^{(\pm)}>=|i_\pm>^{\oplus n}$) it reads

$$\int dU \mathbf{D}(U)^{j_1...j_n}_{i_1...i_n} \mathbf{D}(U^+)^{j_{n+1}...j_{2n}}_{i_{n+1}...i_{2n}} = \mathbf{D}(\Phi_{2n}\Gamma(2n)(\Lambda_n^{(-1)} \otimes \hat{1}_{[n]}))^{\{j^{\oplus 2n}\}}_{\{i^{\oplus 2n}\}}, \tag{3.11}$$

where $\hat{1}_{[n]}$ denotes the 'unity'-permutation of the $S(n)$ group, $\Lambda_n^{(-1)} \in S(n)$ is defined by eq. (3.9), while $\Gamma(2n) = \sum_{\sigma \in S(n)} (\sigma^{-1} \otimes \sigma)$, i.e.

$$\mathbf{D}(\Gamma(2n))^{\{j^{\oplus 2n}\}}_{\{i^{\oplus 2n}\}} = \sum_{\sigma \in S(n)} \mathbf{D}(\sigma^{-1})^{j_1 \dots j_n}_{i_1 \dots i_n} \otimes \mathbf{D}(\sigma)^{j_{n+1} \dots j_{2n}}_{i_{n+1} \dots i_{2n}} \ \in \ S(2n) \ . \tag{3.12}$$

To restore the ρ-labels, $n \to n_\rho$, observe first that the ordering of the $\{U_\rho\}$-factors in eq. (3.4) is associated to the following basis

$$|\tilde{I}_{4n(+)}> = \bigotimes_{\rho=1}^{D} |I_{2n(\rho)}> \quad ; \quad |I_{2n(\rho)}> = |I^{(+)}_{n(\rho)}> \otimes |I^{(-)}_{n(\rho)}>, \tag{3.13}$$

$$|I^{(\pm)}_{n(\rho)}> = \bigotimes_{\nu \neq \rho}^{D-1} |I^{(\pm)}_{n(\rho\nu)}> \ ; \ |I^{(\pm)}_{n(\rho\nu)}> = |i_\pm(\rho)>^{\oplus n(\rho\nu)} \ |i_\pm(\nu)>^{\oplus n(\rho\nu)}, \tag{3.14}$$

where $2n_+ = \sum_{\rho=1}^{D} n_\rho$, and $|I_n^{(\pm)}> = |i_\pm>^{\oplus n}$ (used in eq. (3.11)) matches with $|I^{(\pm)}_{n(\rho)}>$. Therefore, for a given link ρ, the left and the right $S(n_\rho)$-subblocks of $\Gamma(2n_\rho)$ in eq. (3.12) act respectively on $|I^{(+)}_{n(\rho)}>$ and on $|I^{(-)}_{n(\rho)}>$. The same convention is used for the $S(n_\rho)$-subblocks in the direct product $(\Lambda_{n_\rho}^{(-1)} \otimes \hat{1}_{[n_\rho]})$ entering eq. (3.11).

The remaining $S(2n_\rho)$-operator Φ_{2n_ρ}, being considered in the alternatively ordered basis $|\tilde{I}_{2n(\rho)}>$ for each $|I_{2n(\rho)}>$-subsector,

$$|I_{2n(\rho)}> \longrightarrow |\tilde{I}_{2n(\rho)}> = (|i_+(\rho)> \otimes |i_-(\rho)>)^{\oplus n_\rho} \tag{3.15}$$

(with $|i_\pm(\rho)>^{\oplus n(\rho)} = \otimes_{\nu \neq \rho}^{D-1} |i_\pm(\rho)>^{\oplus n(\rho\nu)}$), takes the simple form of the outer product of the 2-cycle permutations $c_2 \in C(2)$

$$\Phi_{2n_\rho} = (c_2)^{\oplus n_\rho} \in S(2n_\rho) \quad ; \quad c_2 : \{12\} \to \{21\} \ , \tag{3.16}$$

where each $c_2 \in S(2)$ acts on the 'elementary' sector $|i_+(\rho)> \otimes |i_-(\rho)>$. It completes the embedding of the $S(2n_\rho)$ operators (3.11), representing the individual 1-link integrals, to act in the common 'enveloping' $S(4n_+)$-space. (It is noteworthy [25] that the three $S(2n_\rho)$ subblocks of the inner-product in the r.h.s. of eq. (3.11) commute with each other.)

3.2 The Dual form of the TEK action.

Let us now turn to the derivation of the master-element $\Xi_{4n_+}(\{R_{\mu\nu}\})$ entering the synthetic representation (3.3) of the master-integral (2.5). For this purpose, we first specify an alternative, more suitable $S(4n_+)$ basis

$$|I_{4n(+)}>=\bigotimes_{\mu\nu=1}^{D(D-1)/2}|I_{4n(\mu\nu)}> \ , \tag{3.17}$$

$$|I_{4n(\mu\nu)}>=(|i_+(\mu)>|i_+(\nu)>|i_-(\mu)>|i_-(\nu)>)^{\oplus n_{\mu\nu}} \tag{3.18}$$

where in eq. (3.18) the product of the four elementary blocks $|i_\pm(\rho)>$ is associated to the elementary $\mu\nu$-holonomy $U_{\mu\nu}$ entering eq. (2.5). As it is derived in Appendix A, in this basis one obtains

$$\Xi_{4n_+}(\{R_{\mu\nu}\})=\bigotimes_{\mu\nu=1}^{D(D-1)/2}P_{4n_{\mu\nu}}(R_{\mu\nu})\cdot\Psi_{4n_{\mu\nu}} \quad ; \quad n_+=\sum_{\{\mu\nu\}}n_{\mu\nu} \ , \tag{3.19}$$

where each of the operators $P_{4n_{\mu\nu}}(R_{\mu\nu})$, $\Psi_{4n_{\mu\nu}}\in S(4n_{\mu\nu})$ is supposed to act on the corresponding $|I_{4n(\mu\nu)}>$ subspace of $|I_{4n(+)}>$. Given (3.18), $\Psi_{4n_{\mu\nu}}$ assumes the simple form of the outer product

$$\Psi_{4n_{\mu\nu}}=(c_4)^{\oplus n_{\mu\nu}}\in S(4n_{\mu\nu}) \quad ; \quad c_4:\ \{1234\}\to\{4123\} \ , \tag{3.20}$$

with each individual 4-cycle permutation $c_4\in C(4)$ acting on the elementary plaquette subspace

$$|i_+(\mu)>|i_+(\nu)>|i_-(\mu)>|i_-(\nu)> \tag{3.21}$$

ordered in accordance with the original pattern (2.5) of the plaquette-labels of $U_{\mu\nu}$.

As for $P_{4n_{\mu\nu}}(R_{\mu\nu})$, making use of the alternative basis $|I_{4n(\mu\nu)}>\to|\tilde{I}_{4n(\mu\nu)}>$ of the $S(4n_{\mu\nu})$-subspace in (3.18):

$$|\tilde{I}_{4n(\mu\nu)}>=|i_+(\mu)>^{\oplus n_{\mu\nu}}|i_+(\nu)>^{\oplus n_{\mu\nu}}|i_-(\mu)>^{\oplus n_{\mu\nu}}|i_-(\nu)>^{\oplus n_{\mu\nu}} \ , \tag{3.22}$$

we employ (see Appendix A) the following representation

$$P_{4n_{\mu\nu}}(R_{\mu\nu})=\hat{1}_{[n_{\mu\nu}]}\otimes\hat{1}_{[n_{\mu\nu}]}\otimes(C_{R_{\mu\nu}}\sqrt{d_{R_{\mu\nu}}})\otimes(C_{R_{\mu\nu}}\sqrt{d_{R_{\mu\nu}}}) \ , \tag{3.23}$$

where $\hat{1}_{[n_{\mu\nu}]}$ denotes the $S(n_{\mu\nu})$-unity. More explicitly, the four (ordered) $S(n_{\mu\nu})$-factors in eq. (3.23) are postulated to act on the corresponding four (ordered) $n_{\mu\nu}$-dimensional subspaces (3.22) of $|\tilde{I}_{4n_{\mu\nu}}>$. Altogether, we have formulated the required (for the derivation of eq. (3.5)) *embedding* of the $S(4n_{\mu\nu})$ operators $\Psi_{4n_{\mu\nu}}$, $P_{4n_{\mu\nu}}(R_{\mu\nu})$ into the *enveloping* $S(4n_+)$-space.

3.3 $B(\{R_{\mu\nu}\})$ as the Tr_{n_+}-character.

Combining together eqs. (3.19) and (3.11) in compliance with the pattern of eq. (3.3), we arrive at the *dual* representation of the master-integral (2.5) as the $S(4n_+)$ *character* (3.5) with the master-element

$$J_{4n_+}(\{R_{\mu\nu}\}) = \Big(\otimes_{\mu\nu=1}^{D(D-1)/2}\ \Psi_{4n_{\mu\nu}}\Big) \cdot \Big(\otimes_{\rho=1}^{D}\Delta_{2n_\rho}(\{R_{\rho\nu}\})\Big)\,, \qquad (3.24)$$

$$\Delta_{2n_\rho}(\{R_{\nu\rho}\}) = \Phi_{2n_\rho} \cdot \Gamma(2n_\rho) \cdot K_{2n_\rho}(\{R_{\nu\rho}\}) \quad \in \quad S(2n_\rho)\ . \qquad (3.25)$$

For later convenience, we have introduced the $S(n_\rho) \bigotimes S(n_\rho)$ combination that in the $|I_{2n(\rho)}>$ basis (3.13) reads

$$K_{2n_\rho}(\{R_{\nu\rho}\}) = \Lambda_{n_\rho}^{(-1)} \bigotimes \Big(\otimes_{\nu\neq\rho}^{D-1} C_{R_{\rho\nu}}\sqrt{d_{R_{\rho\nu}}}\Big)\ , \qquad (3.26)$$

where $\Lambda_{n_\rho}^{(-1)}$ acts onto $|I_{n(\rho)}^{(+)}>$, while each of the $C_{R_{\rho\nu}}$ factors acts on the $|I_{n(\rho\nu)}^{(-)}>$-subspace of $|I_{n(\rho)}^{(-)}>$.

Next, let us complete the duality transformation trading the intermediate $S(4n_+)$ representation (3.5) for its final $S(n_+)$ pattern (2.6). To this aim, it is convenient to start with the alternative following alternative form of the master-element (3.24). As it is demonstrated in Appendix A, the dual representation (3.5) does *not* alter when in the element (3.24)/(3.25) one makes the substitution

$$\otimes_{\rho=1}^{D}\Phi_{2n_\rho} \to \otimes_{\rho=1}^{D}(\Phi_{2n(\rho)})^2 \cong \otimes_{\rho=1}^{D}\hat{1}_{[2n_\rho]} = \hat{1}_{[4n_+]}\ . \qquad (3.27)$$

so that (inside the Tr_{4n_+}-character) in eq. (3.25) all the operators $\Phi_{2n(\rho)}$ can be omitted.

3.3.1 The $D = 2$ case.

Next, it is appropriate to proceed with the simplest $D = 2$ case of (3.5) corresponding to the well studied continuous *SU(N)* gauge theory on a 2-torus. It will provide not only with a cross-check of our $D \geq 2$ formalism but also with the motivation for the announced reduction, $S(4n_+) \longrightarrow S(n_+)$, of the enveloping space. In $D = 2$, the this reduction is encoded in the basic property (following from eqs. (A.1),(A.3) in Appendix A) of the Ψ_{4n} operator (3.20)

$$Tr_n[U_{\mu\nu}^{\oplus n}] = Tr_{4n}[\mathbf{D}(\Psi_{4n})\tilde{U}_{\mu\nu}^{\oplus n}] \quad ; \quad \tilde{U}_{\mu\nu} = U_\mu \otimes U_\mu^+ \otimes U_\nu \otimes U_\nu^+ \ , \qquad (3.28)$$

while $U_{\mu\nu} = U_\mu \cdot U_\nu \cdot U_\mu^+ \cdot U_\nu^+$. Making the substitution $U_\rho \rightarrow \sigma_\rho^{(+)}$, $U_\rho^+ \rightarrow \sigma_\rho^{(-)}$, one obtains

$$Tr_{4n}[\mathbf{D}(\left(\bigotimes_{\rho=1}^{2}(\sigma_\rho^{(+)} \otimes \sigma_\rho^{(-)})\right) \cdot \Psi_{4n})] = Tr_n[\mathbf{D}(\prod_{\rho=1}^{D} \sigma_\rho^{(+)} \cdot \prod_{\mu=1}^{D} \sigma_\mu^{(-)})], \qquad (3.29)$$

where in the l.h.s. the operators $\sigma_\rho^{(\pm)} \in S(n)$ (combined into the *outer* product) act on the associated $|I_n^{(\pm)}>$ subspaces of the $S(4n)$ basis (3.13). As for the ordering inside the two *inner* ρ-products in the r.h.s. of eq. (3.29), in both products it complies with the ordering of the $|i_+(\rho)>$ (or, equally, $|i_-(\rho)>$) elementary blocks in eq. (3.21).

Let us apply the identity (3.29) to the $D = 2$ option of (3.5), (3.24). Combining eq. (3.29) with the orthonormality $P_{R_1}P_{R_2} = \delta_{R_1,R_2}P_{R_1}$ of $P_R = d_R C_R$ and taking into account the standard relation $Tr_n[C_R\sigma] = dimR\chi_R(\sigma)/d_R$ (between the projected Tr_n-trace and the canonical $S(n)$ character χ_R, see e.g. [25]), one easily obtains

$$B(R) = \frac{1}{dimR} \{ \left(\frac{d_R}{n!}\right)^2 \sum_{\{\sigma_\rho \in S(n)\}} \frac{\chi_R([\sigma_1, \sigma_2])}{d_R} \} = \frac{1}{dimR} \ , \qquad (3.30)$$

where $R \in Y_n^{(N)}$, and $[\sigma_1, \sigma_2]$ conventionally stands for $(\sigma_1\sigma_2\sigma_1^{-1}\sigma_2^{-1})$. We have also used that the block in the curly brakets of (3.30) is equal to unity according to the identity derived in [13]. Together with eq. (2.4), the expression (3.30) for $B(R)$ precisely matches with the (genus one) result of [11, 19] derived by the combinatorial method of [20].

3.3.2 The $D \geq 3$ case.

Returning to the generic $D \geq 3$ Tr_{4n_+}-character (2.5) of the master-element (3.24), the $S(4n_+) \to S(n_+)$ reduction of the enveloping space is performed with the help of the following generalization of the $D = 2$ identity (3.29)

$$Tr_{4n_+}[\ \mathbf{D}(\left(\bigotimes_{\mu\nu=1}^{D(D-1)/2} \Psi_{4n_{\mu\nu}}\right) \cdot \left(\bigotimes_{\rho=1}^{D} (\sigma_\rho^{(+)} \otimes \sigma_\rho^{(-)})\right))\ ] =$$

$$= Tr_{n_+}[\ \mathbf{D}(\left(\prod_{\rho=1}^{D} (\sigma_\rho^{(+)} \otimes \hat{1}_{[\frac{n_+}{n_\rho}]})\right) \cdot \left(\prod_{\rho=1}^{D} (\sigma_\rho^{(-)} \otimes \hat{1}_{[\frac{n_+}{n_\rho}]})\right))\ ]. \qquad (3.31)$$

Let us simply explain the meaning of the above pattern, while for more details see Appendix A. In the l.h.s. of (3.31), the *outer* $\mu\nu$-product is defined in the same way as in eq. (3.24), and the operators $\sigma_\rho^{(\pm)} \in S(n_\rho)$ (composed into the *outer* ρ-product) act on the associated $|I_{n(\rho)}^{(\pm)}>$ subspace of the $S(4n_+)$ basis (3.13). As for the r.h.s. of (3.31), we first construct the following $S(n_+)$ basis. To begin with , one is to introduce the N-dimensional spaces $|i(\mu\nu)>$, $i = 1, ..., N,$ parametrized by the plaquette label $\mu\nu = 1, ..., D(D-1)/2$. Then, the $S(n_+)$ operators (recall that $n_+ = \sum_{\{\mu\nu\}} n_{\mu\nu}$) can be viewed as acting on

$$|I_{n(+)}>= \bigotimes_{\mu\nu=1}^{D(D-1)/2} |I_{n(\mu\nu)}> \quad , \quad |I_{n(\mu\nu)}>= (|i(\mu\nu)>)^{\oplus n_{\mu\nu}} \ , \qquad (3.32)$$

according to the same rule (3.10) that has been already used for the $S(4n_+)$ operators. Given this convention, each operator $\sigma_\rho^{(\pm)}$ is postulated to act on the associated $S(n_\rho)$ subspace $|\tilde{I}_{n(\rho)}>$ of $|I_{n(+)}>$,

$$|\tilde{I}_{n(\rho)}>= \bigotimes_{\nu\neq\rho}^{D-1} (|i(\rho\nu)>)^{\oplus n_{\rho\nu}} \quad , \quad n_\rho = \sum_{\nu\neq\rho}^{D-1} n_{\rho\nu} \ , \qquad (3.33)$$

where the ordering of the $|i(\rho\nu)>$ blocks matches with the one in eq. (3.32). As for $\hat{1}_{[n_+/n_\rho]}$, in eq. (3.31) it denotes the unity permutation on the $S(n_+ - n_\rho)$ subspace of (3.32) complementary to (3.33). In the $D = 2$ case, where $n_+ = n_1 = n_2$, the general eq. (3.31) readily reduces to its degenerate variant (3.29).

To complete the construction (3.31), one should specify the ordering inside the two inner ρ-products of its r.h. side. We defer this task till the end of the section and now apply the identity (3.31) to the Tr_{4n_+} character (3.5) of the master-element (3.24). For this purpose, all what we need is the proper identification of the composed into $J_{4n_+}(\{R_{\mu\nu}\})$ permutations with $\sigma_\rho^{(\pm)}$. To this aim, let denote by $\lambda_{\mu\nu}$ and λ_ρ the permutations which enter the definition (3.1) of the relevant operators $C_{R_\phi} = P_{R_\phi}/d_{R_\phi} \in S(n_\phi)$ (combined into K_{2n_ρ} of eq. (3.26), with $\Lambda_{n_\rho}^{(-1)} \in S(n_\rho)$ being given by eq. (3.9)) assigned with the associated labels $\phi \in \{\mu\nu\}, \{\rho\}$. Complementary, let σ_ρ stands for the permutations entering the definition (3.12) of $\Gamma(2n_\rho)$. Identifying $(\otimes_{\mu\neq\rho}^{D-1}\lambda_{\mu\rho}) \cdot \sigma_\rho \to \sigma_\rho^{(-)}$, $\lambda_\rho \cdot \sigma_\rho^{-1} \to \sigma_\rho^{(+)}$ (without numb summation over ρ), after some routine machinery one derives for the master-element

$$A_{n_+}(\{R_{\mu\nu}\}) = \prod_{\rho=1}^{D} \sum_{R_{n_\rho} \in Y_{n_\rho}^{(N)}} \frac{d_{R_\rho}^2}{n_\rho! dim R_\rho} \sum_{\sigma_\rho \in S(n_\rho)} F(\{\sigma_\rho\}; \{R_\phi\}) \ , \tag{3.34}$$

$$F = \left(\bigotimes_{\{\mu\nu\}} C_{R_{\mu\nu}}\right) \cdot \left(\prod_{\{\rho\}} (\sigma_\rho \otimes \hat{1}_{[\frac{n_\pm}{n_\rho}]})\right) \cdot \left(\prod_{\{\lambda\}} (\sigma_\lambda^{-1} C_{R_\lambda} \otimes \hat{1}_{[\frac{n_\pm}{n_\lambda}]})\right) , \tag{3.35}$$

so that its trace (2.6) determines the $B(\{R_{\mu\nu}\})$-block (2.5) of the TEK partition function (PF).

Altogether, it establishes the exact duality transformation of the TEK PF (2.1) which is one of the main results of the paper. In certain sense (modulo the explicit presence of the σ_ρ-twists), it provides with the $D \geq 3$ generalization of the representation [11, 19] for the PF of the continuous gauge theory on a $2d$ manifold. Also, it can be compared with the considerably simpler pattern of the PF of the judiciously constructed solvable D-matrix models [24] where $B(\{R_{\mu\nu}\})$ depends nontrivially only on the associated generalized Littlewood-Richardson coefficients. (On the contrary, the pattern (3.24) encodes the general Klebsch-Gordan coefficiens.)

3.3.3 The ordering inside the inner ρ-products .

Finally, let us discuss the ordering inside the two inner ρ-products of the r.h. side of eq. (3.31). As it is shown in Appendix A, this ordering is entirely predetermined by the following characteristics $G(\rho)$ of a particular ρ-label

called its *cardinality*. Consider the $D(D-1)/2$ dimensional vector $\mathbf{M}(\{\rho\})$ defined so that its components $M_k(\{\rho\})$ are in one-to-one correspondence with the $D(D-1)/2$ labels $\mu\nu$ (where $1 \leq \mu < \nu \leq D$). The latter parametrize the $\mu\nu$ th plaquette-holonomies $U_{\mu\nu}$ entering the characters in eq. (2.5). Let the $\mu\nu$ th component of $\mathbf{M}(\{\rho\})$ is equal to the first link-label μ of the plaquette-label: $M_{\mu\nu}(\{\rho\}) = \mu$. Then, the cardinality (ranging from 0 to $D-1$) is postulated to be $G(\mu) = \sum_{k=1}^{D(D-1)/2} \delta[\mu, M_k(\{\rho\})]$, i.e. the number of times the particular μ-label enters the entries of the vector $\mathbf{M}(\{\rho\})$. In eq. (2.5), it is always possible to arrange for a *nondegenerate* cardinality assignement $\{G(\rho)\}$ when $G(\rho_1) \neq G(\rho_2)$ if $\rho_1 \neq \rho_2$. Then, *given a nondegenerate set* $\{G(\rho)\}$, *the* $(\sigma_\rho^{(\pm)} \otimes \hat{1}_{[n_+/n_\rho]})$ *factors in the eq. (3.31) are ordered (from the left to the right) according to the successively decreasing* $G(\rho)$*-assignements of their* ρ*-labels.*

4 Schur-Weyl transformation of $B(\{n_{\mu\nu}\})$.

To transform the dual representation (2.6)/(3.34) of the TEK PF $\tilde{X}_D$ into the $D \geq 2$ stringy representation like (1.9), one is rewrite $B(\{n_{\mu\nu}\})$ entirely in terms of the symmetric groups' variables which are suitable for the algebraic definition of the topological data associated to the mappings (1.11). For this purpose, we employ the following two useful identities (derived in Appendices B and D) which reflect the Schur-Weyl complementarity of the Lie and the symmetric groups. The first one trades the ubiquitous operator (3.9) for the product of the two elements of the associated $S(n)$ algebra

$$\Lambda_n^{(m)} = P_n^{(N)} \cdot (N^n \Omega_n)^m \quad , \quad P_n^{(N)} = \sum_{R \in Y_n^{(N)}} P_R \,, \tag{4.1}$$

where the projector $(P_n^{(N)})^2 = P_n^{(N)}$, $P_n^{(N)} = 1$ if $n < N$, (that independently appeared within the method of [27]) reduces the $S(n)$ Y_n-variety of irreps to the $Y_n^{(N)}$-one of either *U(N)* or $SU(N)$. As for the second element

$$\Omega_n = \sum_{\sigma \in S(n)} (1/N)^{n-K_{[\sigma]}} \, \sigma \quad ; \quad [\, \Omega_n, \rho \,] = 0 \;, \; \forall \rho \in S(n), \tag{4.2}$$

(belonging the center of the $S(n)$-algebra), it is defined [13] by the equation

$$(dimR)^m = \frac{\chi_R((N^n\Omega_n)^m)}{d_R}\left(\frac{d_R}{n!}\right)^m \quad ; \quad n - K_{[\sigma]} = \sum_{k=1}^{n}(k-1)p_k \ , \tag{4.3}$$

where $m \in \mathbf{Z}$, and the factor $K_{[\sigma]} = \sum_{k=1}^{n} p_k$ in eq. (4.2) denotes the total number of various k-cycles in the cyclic decomposition of the conjugacy class $[\sigma] = [1^{p_1}, 2^{p_2}, ..., n^{p_n}]$, $\sum_{k=1}^{n} kp_k = n$.

Generalizing (4.1), the second key-identity deals with the similar sum weighted this time by the factor $e^{-\Gamma}$ (which defines a generic model (1.1))

$$\sum_{R\in Y_n^{(N)}} e^{-\Gamma(..,\{C_p(R)\})} \ d_R \left(\frac{n!dimR}{d_R}\right)^m C_R = P_n^{(N)} \cdot (N^n\Omega_n)^m \cdot Q_n(\Gamma). \tag{4.4}$$

In the Heat-Kernal case (1.7) where $\Gamma(..) = \lambda C_2(R)/2N$, the $S(n)$-algebra valued operator $Q_n(\Gamma)$ reads

$$Q_n(\Gamma) = exp[-\frac{\lambda}{2}(n - \frac{n^2}{N^2})] \ exp[-\frac{\lambda}{N}\hat{T}_2^{(n)}] \quad , \quad \hat{T}_2^{(n)} = \sum_{\tau\in T_2^{(n)}} \tau \ , \tag{4.5}$$

where $T_2^{(n)} \equiv T_2$ denotes the $S(n)$ conjugacy class $[1^{n-2}2^1]$ of the simple transposition, and $\lambda = \tilde{g}^2 N$. In a generic admissible model (1.1), as it is demonstrated in Appendix E, the operator $Q_n(\Gamma)$ generalizes to

$$Q_n(\Gamma) = exp[\sum_{\{p\}} \upsilon_{\{p\}}(\{\tilde{b}_k\}, n, N) \ \hat{T}_{\{p\}}^{(n)}] \quad , \quad \hat{T}_{\{p\}}^{(n)} = \sum_{\xi^{\{p\}}\in T_{\{p\}}^{(n)}} \xi^{\{p\}}, \tag{4.6}$$

where $\hat{T}_{\{p\}}^{(n)}$ denotes the sum of the $S(n)$ permutations belonging to a particular conjugacy class $T_{\{p\}}^{(n)}$ labelled by the partition $\{p\}$ of n: $\sum_{i=1}^{n} kp_k = n, .$ As for the weight $\upsilon_{\{p\}}(..)$, it assumes the form (which, in particular, results in the required asymptotics (1.6))

$$\frac{\upsilon_{\{p\}}(\{\tilde{b}_k\}, n, N)}{N^{-\sum_{k=1}^{n}(k-1)p_k}} = \sum_{m=0}^{M_{\{p\}}} \sum_{l\geq[m/2]} s_{\{p\}}(\{\tilde{b}_k\}, m, l) \ N^{-2l} \ n^m, \tag{4.7}$$

where $m, l \in \mathbf{Z}_{\geq 0}$, and $[m/2] = m/2$ or $(m+1)/2$ depending on whether m is even or odd (while $s_{\{p\}}(..) \sim O(N^0)$). The specific pattern of $\upsilon_{\{p\}}(..)$

implies that in eq. (1.1) the function $\Gamma(..)$ satisfies certain conditions (see Appendix E) that ensure the consistent stringy interpretation of (4.7) to be discussed in Section 6 (for eq. (4.5)) and in Appendix E (for eqs. (4.6)/(4.7)).

Combining (4.1),(4.4) with the delta-function reformulation (2.8) of the Tr_{n_+} character (derived in Appendix B), we arrive at the explicit $\otimes_\phi S(n_\phi)$ representation for the building block $B(\{n_{\mu\nu}\})$ of the TEK partition function (2.4). This central, for the present discussion of the lattice Gauge String, expression reads

$$\delta_{n_+}(\Lambda_{n_+}\left(\bigotimes_{\{\mu\nu\}}\frac{Q_{n_{\mu\nu}}\Lambda_{n_{\mu\nu}}}{n_{\mu\nu}!}\right)\sum_{\{\sigma_{\tilde{\rho}}\}}\prod_{\{\rho\}}(\sigma_\rho\otimes\hat{1}_{[\frac{n_+}{n_\rho}]})\prod_{\{\mu\}}(\sigma_\mu^{-1}\Lambda^{(-1)}_{n_\mu}\otimes\hat{1}_{[\frac{n_+}{n_\mu}]})), \quad (4.8)$$

where, to all orders in *1/N*, one can safely use the *SU(N)* variant of the representation (4.1) of $\Lambda^{(m)}_{n_\phi}$. It is noteworthy that the $[\mathbf{Z_2}]^{\oplus D(D-1)/2}$ invariance (with respect to $R_{\mu\nu}\leftrightarrow\bar{R}_{\mu\nu}$) of the sums in (2.1) defining the plaquette-factor (1.1) results in the invariance of eq. (4.8) under the simultaneous permutations $\rho,\mu\to\sigma(\rho),\sigma(\mu)$, $\forall\sigma\in S(D)$ of the link-labels $\rho,\mu=1,...,D$, in the two ordered inner products.

5 The stringy form of $B(n)$ in $D=2$.

To begin with, in the $D=2$ case (where $n_+=n_{12}=n_1=n_2$) all the involved into (4.8) $S(n_\phi)$ operators act in one and the same $S(n)$-space that matches with the reduced formula (3.29). The resulting amplitude in the Heat-Kernal case (1.7) reads (with $B(0)\equiv 1$)

$$B(n)=\frac{e^{-\frac{\lambda}{2}(n-\frac{n^2}{N^2})}}{n!}\sum_{\{\sigma_\rho\},T_2^{(n)}\in S(n)}\delta_n(P_n^{(N)}e^{-\frac{\lambda}{N}\hat{T}_2^{(n)}}(N^n\Omega_n)^{1-2+1}[\sigma_1,\sigma_2]) \quad (5.1)$$

that is in complete agreement with the genus-one result [27] of Baez and Taylor derived by a different method. It provides with the more compact reformulation of the Gross-Taylor stringy pattern (1.9), although the transformation (to be summarized by eqs. (5.7),(5.8) below) relating the two representations is not entirely manifest. Let us proceed recasting (5.1) into the form 'almost' equivalent to the one of (1.9).

To make contact with the pattern of the f-mapping of eq. (1.9), first it is convenient to decompose

$$P_n^{(N)} = \sum_{T_{\{p\}}^{(n)} \in S(n)} P_n^{(N)}(T_{\{p\}}^{(n)}) \ \hat{T}_{\{p\}}^{(n)} \ , \tag{5.2}$$

where $\hat{T}_{\{p\}}^{(n)}$ is defined in eq. (4.6). Then, expanding all the exponents except $e^{-\frac{\lambda}{2}n} = e^{-\tilde{\sigma}_0 n}$, one is to rewrite (5.1) as

$$B(n) = \sum_{i,s,t \geq 0} \sum_{T_{\{p\}} \in S(n)} \ \sum_{f \in \tilde{M}} \frac{N^{-2(t+s)-i}}{|C_f(\{p\})|} P_n^{(N)}(T_{\{p\}}) \ K_n(i,s,t), \tag{5.3}$$

$$K_n(i,s,t) = e^{-\frac{\lambda}{2}n} \ \frac{(\lambda)^{i+s+t}}{i!s!t!} \ \frac{(-1)^i n^s (n^2-n)^t}{2^{s+t}} \ , \tag{5.4}$$

where in the first exponent of (5.1) one is to decompose $n^2 = n/2 + n(n-1)/2$, and for simplicity we denote $T_{\{p\}} \equiv T_{\{p\}}^{(n)}$ in the rest of the section. As for the symmetry factor,

$$\sum_{f \in \tilde{M}(\{p\},n,i)} \frac{1}{|C_f(\{p\})|} = \sum_{\{\sigma_\rho \in S(n)\}} \frac{1}{n!} \ \delta_n(\ \hat{T}_{\{p\}} \ (\hat{T}_2)^i \ [\sigma_1, \sigma_2] \ ), \tag{5.5}$$

it emerges when one reformulates the r.h.s. of (5.5) as the sum over certain maps (1.9) (to be explicitly constructed below). The latter can be viewed as the topological mappings (i.e. immersions *without* singularities)

$$f \ : \qquad f(\tilde{M} - \{f^{-1}(q_s)\}) = M - \{q_s\} \ . \tag{5.6}$$

of the space $\tilde{M} - \{f^{-1}(q_s)\}$ onto the base-space torus M with $i+1$ deleted points $\{q_s\}$. These maps define [26] the admissible (by the data in the r.h.s. of (5.5)) branched covering spaces $\tilde{M}$ of M which can be visualized as the Riemann surfaces $\tilde{M} \equiv \tilde{M}(\{p\},n,i)$ (to be identified with the worldsheets of the *YM*-flux) with n-sheets and $i+1$ branch points located at $\{q_s\}$.

As for the sums (5.3) over the nonnegative integers i,t,s, in addition to the number of the 'movable' simple branch-points (where two sheets are identified), they refer to the extra 'movable' singularities of the map (1.9). Namely, one is to attach to $\tilde{M}(\{p\},n,i)$ the t collapsed to a point microscopic tubes (connecting two sheets) and the s collapsed to a point

handles (glued to a single sheet). Altogether, it results in the worldsheet $\tilde{M}(\{p\}, n, i|t, s)$.

The important observation [13] is that the factor $|C_f(\{p\})|$ is equal to the number of distinct automorphisms of the branched covering space $\tilde{M}(\{p\}, n, i|t, s)$ in question. Complementary, in the absence of the $P_n^{(N)}$-twist (i.e. when $T_{\{p\}} \to \hat{1}$), the *1/N* factor enters (5.3) in the power equal to the $G = 1$ option of the Riemann-Hurwitz formula $h = n{\cdot}(2G-2)+2(t+s)+i$ calculating the overall genus of the corresponding (modified) Riemann surface $\tilde{M}(\hat{1}, n, i|t, s)$. Also, in what follows, we assume that that the contribution of the 'movable' collapsed handles is reabsorbed into the redefinition $\tilde{\sigma}_0 = \lambda/2 \to \lambda(1-1/N^2)/2$ of the $SU(N)$ bare string tension which eliminates the corresponding singularities of the map (1.11). (In the case (4.6) of the generic model (1.1), the modified tension is given by the $n = 1$ restriction of $-\upsilon_{\{p\}}(\{\tilde{b}_k\}, n, N)$ associated to $\hat{T}^{(n)}_{\{p\}} = \hat{1}_{[n]}$.)

¿From the general expression (4.6), it is clear that the pattern (1.9) emerges in a generic model (1.1) as well. In particular, owing to the pattern of eq. (4.7), the large N asymptotics $-ln[Q_n(\Gamma)] = n\tilde{\sigma}_0(\{\tilde{b}_k\})(1 + O(1/N))$ is consistently provided by the $\hat{T}_{\{p\}} = \hat{1}$ term of (4.6). For $T_{\{p\}} \neq \hat{1}$, the leading $l = 0$ term in eq. (4.7) describes the branch-point canonically parametrized (see e.g. [18]) by $T_{\{p\}}$. The latter point decreases the associated Euler character by $\sum_k (k-1)p_k$ which matches (akin to the pattern (4.3) of Ω_n) with power of the *1/N* factor assigned in (4.7) to $\hat{T}_{\{p\}}$. As it is discussed in Appendix E (where the earlier results [14, 18] are summarized and reformulated), the $l \geq 1$ terms can be reinterpreted as the movable subsurfaces (of various topologies) collapsed to a point.

As for the factor $P_n^{(N)}(T_{\{p\}})$, inherited from the decomposition (5.2) of the projector $P_n^{(N)}$, its dependence on N is not particularly suitable for a manifest $1/N$ expansion like (1.9). It calls for a nontrivial resummation which would reproduce the well-defined large N *SC* series (1.9) (obtained by Gross and Taylor without resort to (5.1)). The latter pattern effectively eliminates $P_n^{(N)}(T_{\{p\}})$ at the expense of working with the $S(n^+) \otimes S(n^-)$ double-representation, $B(n) \to B(\{n^\pm\})$, that refers to the two coupled sectors of the opposite worldsheet orientation. The prescription to reconstruct $B(\{n^\pm\})$ from the amplitude like (4.1) is quite simple: all the involved $S(n)$-

structures are traded for their $S(n^+)\otimes S(n^-)$ counterparts

$$\delta_n(..)\to\delta_{n^+\times n^-}(..)\quad,\quad N^n\Omega_n\to N^{n^++n^-}\Omega_{n^+,n^-}\quad,\quad \sigma_\rho\to\sigma^+_\rho\otimes\sigma^-_\rho\,, \tag{5.7}$$

$$Q_n\to Q_{n^+,n^-}=e^{-\frac{\lambda}{2}(n^++n^--((n^+)^2+(n^-)^2-2n^+n^-)/N^2)}e^{-\frac{\lambda}{N}(\hat{T}_2^{(n^+)}+\hat{T}_2^{(n^-)})}, \tag{5.8}$$

where eq. (5.8) imples the Heat-Kernal case (1.7) and can be generalized to a generic model (1.1). (The definition and interpretation of Ω_{n^+,n^-} and other ingredients in eqs. (5.7),(5.8) can be found in [13].)

5.1 Construction of the branched covering spaces.

In the remaining subsections, we discuss the major issues related to the effective enumeration of the mappings (5.6) and their automorphisms. Let us proceed with an explicit algorithm which, given the symmetric group data in the r.h.s. of (5.5), reconstructs the topology of the Riemann surfaces $\tilde{M}(\{p\},n,i)$ in the l.h.s. of (5.5). As the elements $\hat{T}_{\{p\}}$, $(\hat{T}_2)^i$ are associated [15] to the branch points (BPs), it is convenient to start with the simpler case of the topological covering spaces (without BP's singularities) removing the latter elements. The full branched covering spaces (BCSs) can be reproduced reintroducing the BPs onto the corresponding covering spaces (CSs).

As for a particular n-sheet CS $\tilde{M}$ of a given $2d$ surface M (the 2-torus in what follows), it can be composed with the help of the cutting-gluing rules borrowed from the constructive topology. To begin with, cut a 2-torus M along the two uncontractible cycles $\alpha(\rho)$ trading the latter for the pairs of edges $\alpha(\rho)\otimes\beta(\rho)$, $\rho=\mu,\nu$. It makes M into a rectangular $H_{\mu\nu}$ with the boundary edge-path represented as $\alpha(\mu)\alpha(\nu)\beta^{-1}(\mu)\beta^{-1}(\nu)$. Then, consider the *trivial* covering $\tilde{H}_{\mu\nu}=H_{\mu\nu}\otimes\Upsilon_n$ (where $\Upsilon_n=\{1,...,n\}$) of $H_{\mu\nu}$ by n copies of this rectagular with the boundary edge-paths given by $\alpha(\mu_k)\alpha(\nu_k)\beta^{-1}(\mu_k)\beta^{-1}(\nu_k)$, $k=1,...,n$. Perform the set of the *pairwise* reidentifications of the involved edges

$$\alpha(\rho_k)=\beta(\rho_{\sigma_\rho(k)})\qquad;\qquad \sigma_\rho\ :\ \ k\to\sigma_\rho(k)\quad,\quad k\in\Upsilon_n\ , \tag{5.9}$$

where $\sigma_\mu\equiv\sigma_1$, $\sigma_\nu\equiv\sigma_2$ are supposed to satisfy the δ_n-constraint (5.5) (with the excluded contribution of $\hat{T}_{\{p\}}(\hat{T}_2)^i$).

Evidently, the two sets (5.9) of the reidentificatrions can be concisely represented as the two *closed* (i.e. without branch end-points) *branch cuts* ϖ_ρ of the Riemann surface $\tilde{M}(\{\sigma_\rho\})$ with n sheets. According to the pattern (5.9), each connected component of $\tilde{M}(\{\sigma_\rho\})$ has the topology of 2-torus. In compliance with (1.9), it matches with the N-independence of the argument of the δ_n-function (5.1) taking place after the exclusion of $P_n^{(N)} exp[-\lambda \hat{T}_2^{(n)}/N]$.

To reintroduce the branch points (encoded in the $\hat{T}_{\{p\}}(\hat{T}_2)^i$ factor of (5.5)), recall that each admissible BP is the end-point q_k of the associated branch cut $\varpi^{(k)}$ which should be included additionally to the closed cuts ϖ_ρ of the CS $\tilde{M}(\{\sigma_\rho\})$. To implement $\varpi^{(k)}$, we first cut $\tilde{M}(\{\sigma_\rho\})$ along the support of $\varpi^{(k)}$. (Both $\varpi^{(k)}$ and ϖ_ρ are all supposed to terminate at a common base-point $p = \alpha(\mu) \cap \alpha(\mu)$ of M.) Then, on the left and on the right sides of each cut $\varpi^{(k)}$, the resulting two copies of the n new edges of the sheets are reidentified according to the prescription (5.9). The only modification is that, instead of σ_ρ, one is to substitute the appropriate permutations $\xi^{\{p\}} \in T_{\{p\}}$ and $\tau^{(s)} \in T_2$ entering respectively $\hat{T}_{\{p\}}$ and the s th $\hat{T}_2$-factor in the inner product $(\hat{T}_2)^i$. It completes the construction of the admissible Riemann surfaces $\tilde{M}(\{p\}, n, i)$ entering the l.h.s. of (5.5).

5.2 The homomorphism of $\pi_1(M - \{q_s\}|p)$ into $S(n)$.

To enumerate the equivalence classes of BCSs and justify the asserted interpretation of $|C_f(\{p\})|$, one is to employ the relation to the following *group homomorphism* [15] where the first homotopy group $\pi_1(M - \{q_s\}\}|p)$

$$\psi \ : \quad \pi_1(M - \{q_s\}|p) \to S(n) \ , \tag{5.10}$$

is mapped into $S(n)$. (In eq. (5.10), $M - \{q_s\}$ denotes the base-surface M with the $i+1$ excluded points q_s, associated to the branch points, and with the base-point p.) Given an BCS encoded in the δ_n-function (5.5), choose a set $\Upsilon_n = \{1, 2, ..., n\}$ to label the n sheets (at the base-point p). Consider the lift [26, 18] of the closed paths in $M - \{q_s\}$ (defining $\pi_1(M - \{q_s\}|p)$) into the covering space $\tilde{M} - \{f^{-1}(q_s)\}$. Then, the (equivalence classes of the) paths induce the permutations of the labels which determine the corresponding $S(n)$ operators acting on Υ_n.

To make (5.10) explicit, let us first specify the pattern of the first homotopy group. Consider a topological space $T - \{q_1, ..., q_m\}$ which, for our later purposes, is allowed to be a (CW) $2d$ cell-complex T (*not* necessarily reduced to a $2d$ surface) with m deleted points q_k. Recall that in this case the group $\pi_1(T - \{q_1, ..., q_m\})$ (in what follows we will everywhere omit the specification of the base-point p) can be represented as the following abstract group [26]. The generators of the latter group are associated to the homotopy equivalence classes (HEC) of the *uncontractible* closed paths based at a given point p (supposed to be distinct from the set $\{q_s\}$). In the case at hand, additionally to the generators $\alpha_r,\ r = 1, ..., P$ (corresponding to P HECs of the uncontractible cycles of T), there are extra m generators $\gamma^{(s)},\ s = 1, ..., m,$ which refer to the HECs of closed paths encircling a single deleted point q_s. Finally, there exists a constructive algorithm to find the complete set of $K \in \mathbf{Z}_{\geq 1}$ *relations* $\{F_l(\{\alpha_r, \gamma^{(s)}\}) = 1\}_{l=1,...,K}$ that completes the intermediate mapping of the $\pi_1(T - \{q_1, ..., q_m\})$ generators into the abstract group.

Returning to the case of a genus g $2d$ surface $T = M_g$, there is a single relation (with $P = 2g,\ [\alpha_i, \alpha_k] \equiv \alpha_i \alpha_k \alpha_i^{-1} \alpha_k^{-1}$) defining $\pi_1(M_g - \{q_s\})$

$$F(\{\alpha_r, \gamma^{(s)}\}) = \left(\prod_{j=1}^{g} \ [\alpha_j, \alpha_{g+j}] \ \prod_{s=1}^{m} \gamma^{(s)} \right) = 1 \ . \tag{5.11}$$

Comparing (5.11) with the pattern (5.1)/(5.5), one deduces the explicit form of the homomorphism (5.10) (with the identification $g = 1,\ m = i + 1$)

$$\psi \ : \ \ \psi(\alpha_\rho) = \sigma_\rho \ ; \ \ \psi(\gamma^{(1)}) = \xi^{\{p\}} \ ; \ \ \psi(\gamma^{(s)}) = \tau^{(s-1)} \ , \ s \geq 2, \tag{5.12}$$

where $\xi^{\{p\}} \in T_{\{p\}}$ and $\tau^{(s-1)} \in T_2$ which enter respectively $\hat{T}_{\{p\}}$ and the $(s-1)$ th $\hat{T}_2$-factor (in the inner product $(\hat{T}_2)^i$).

5.3 The symmetry factor.

Given the homomorphisms (5.12), one can readily enumerate the equivalence classes $\tilde{M}(\{p\}, n, i)$ of the BCSs employing the following notion of equivalence [13] of two homomorphisms ψ_1 and ψ_2. The latter are postulated to belong to the same equivalence class if there exists some $\eta \in S(n)$ so that

$$\psi_1(\zeta) = \eta \ \psi_2(\zeta) \ \eta^{-1} \quad , \quad \forall \zeta \in \pi_1(M - \{q_1, ..., q_m\}) \ \ ; \ \ \eta \in S(n). \tag{5.13}$$

Then, the basic theorem [15, 26] of the topological coverings ensures that the inequivalent homomorphisms (5.12) are in one-to-one correspondence with the associated homeomorphically distinct branched covering spaces.

Finally, let $\kappa: \ f \circ \kappa = f$, denotes a particular automorphism of the branched covering space $\tilde{M}$. Being restricted to the n-set $\Upsilon_n = f^{-1}(\tilde{p})$ ($p \neq q_s, \ \forall s$), the group of the automorphisms is isomorphic [26] to (the conjugacy class of) the $S(n)$-subgroup $C_f(\{p\})$ that induces conjugations (5.13) leaving *all* the images $\psi_1(\zeta)$ *invariant*: $\psi_1(\zeta) = \psi_2(\zeta), \ \forall\zeta, \ \forall\eta(\kappa) \in C_f(\{p\})$. In turn, it justifies [13] the required interpretation of $|C_f(\{p\})|$.

6 The stringy form of $B(\{n_{\mu\nu}\})$ in $D \geq 3$.

To rewrite the $D \geq 3$ amplitude (4.8) in the form generalizing the $D = 2$ stringy pattern (5.3)-(5.5), we first expand each factor $Q_{n_{\mu\nu}}$ and select the $i_{\mu\nu}$ th power $(\hat{T}_2^{(n_{\mu\nu})})^{i_{\mu\nu}}$ (where the $k_{\mu\nu}$ th $\hat{T}_2^{(n_{\mu\nu})}$-factor in the latter product is supposed to be defined via eq. (4.5) in terms of $\tau_{\mu\nu}^{(k_{\mu\nu})} \in T_2^{(n_{\mu\nu})}$, $k_{\mu\nu} = 1, ..., i_{\mu\nu}$). Complementary, akin to eq. (4.6) one is to decompose

$$\Lambda_{n_\phi}^{(m_\phi)} = \sum_{T_{\{p_\phi\}}^{(n_\phi)} \in S(n_\phi)} \Lambda_{n_\phi}^{(m_\phi)}(T_{\{p_\phi\}}^{(n_\phi)}) \sum_{\xi^{\{p_\phi\}} \in T_{\{p_\phi\}}^{(n_\phi)}} \xi^{\{p_\phi\}} \ , \tag{6.1}$$

and separate the contribution of a given $\hat{T}_{\{p_\phi\}}^{(n_\phi)} \equiv \hat{T}_{\{p_\phi\}}$ (defined by eq. (4.6)).

Given the above expansions, one can prove (see below) that the associated building block of (4.8)

$$\sum_{\{\sigma_\rho\}} \delta_{n_+}(\hat{T}_{\{p_+\}} \bigotimes_{\{\mu\nu\}} \hat{T}_{\{p_{\mu\nu}\}} \frac{(\hat{T}_2^{(n_{\mu\nu})})^{i_{\mu\nu}}}{n_{\mu\nu}!} \prod_{\{\rho\}} (\sigma_\rho \otimes \hat{1}_{[\frac{n_+}{n_\rho}]}) \prod_{\{\mu\}} (\sigma_\mu^{-1} \hat{T}_{\{p_\mu\}} \otimes \hat{1}_{[\frac{n_+}{n_\mu}]})) =$$

$$= \sum_{\varphi \in \tilde{M}(\{p_\phi\},\{n_\phi\},\{i_{\mu\nu}\})} \frac{1}{|C_\varphi(\{p_\phi\})|} \tag{6.2}$$

can be rewritten as the sum over the relevant $D \geq 2$ mappings (1.11) to be reconstructed in the next subsection. Extending the $D = 2$ theorem due to Gross and Taylor, each term of the latter sum is weighted by the

inverse number $|C_{\varphi}(\{p_\phi\})|$ of distinct automorphisms associated to a given $\tilde{M}_{\varphi} \equiv \tilde{M}(\{p_\phi\},\{n_\phi\},\{i_{\mu\nu}\})$.

As we will demonstrate, the involved into (6.2) spaces $\tilde{M}_{\varphi}$ can be viewed as the generalized Riemann surfaces to be identified with the worldsheets of the *YM*-flux 'wrapped around' the EK $2d$ cell-complex (2.2). Combining all the pieces together, the stringy reinterpretation of (4.8) essentially follows the steps discussed in the $D=2$ case. Namely, leaving aside the $P^{(N)}_{n_\phi}(T_{\{p_\phi\}})$ factors (4.1), the rest of the ingredients of (4.8) readily fit in the consistent $D \geq 3$ extension (1.11) of the Gross-Taylor stringy pattern (1.9). Indeed, by the same token as in the $D=2$ analysis, the involved (movable or nonmovable) branch points, microscopic tubes, and handles are weighted by the $1/N$- and n_ϕ-dependent factors according to their contribution to the Euler character of $\tilde{M}_{\varphi}$. The latter 'local' matching is completed by eq. (6.2) together with the following 'global' matching. To see it, let us remove temporarily the branch points and the collapsed subsurfaces via the substitution

$$\Lambda^{(m_\phi)}_{n_\phi} \; \rightarrow \; N^{n_\phi m_\phi} \qquad ; \qquad Q_{n_{\mu\nu}} \rightarrow exp[-\lambda n_{\mu\nu}/2] \; , \tag{6.3}$$

into eq. (4.8). Then, the remaining $\lambda n_{\mu\nu}$-dependent factor and the overall power $(1/N)^{\varepsilon}$ (inherited from eq. (4.8))

$$\varepsilon = \sum_{\mu\nu=1}^{D(D-1)/2} n_{\mu\nu} - \sum_{\rho=1}^{D} n_\rho + n_+ = 0 \quad ; \quad 2n_+ = \sum_{\{\rho\}} n_\rho = 2\sum_{\{\mu\nu\}} n_{\mu\nu}, \tag{6.4}$$

match with the area and the 2-tora topology (revealed below) of each connected component of $\tilde{M}_{\varphi}$ corresponding to the deformation (6.3).

By the same token as in the $D=2$ case, the manifest stringy representation of the full amplitude (4.8) (including the $P^{(N)}_{n_\phi}(T_{\{p_\phi\}})$-factors) calls for a resummation into the appropriate large N *SC* series. Presumably, it eliminates these factors trading each remaining in (4.8) $S(n_\phi)$-operator for its $S(n^+_\phi)\otimes S(n^-_\phi)$ descedant. It is suggestive that the short cut way for the latter reformulation is provided by the direct extention of the $D=2$ prescription (5.7),(5.8): one is to substitute the labels n, n^+, n^- by their properly associated ϕ-dependent counterparts $n_\phi, n^+_\phi, n^-_\phi$ (where n_ϕ, $\phi \in \{\mu\nu\}, \{\rho\}, +$, labels the relevant $S(n_\phi)$-structures). Note also that, similarly to the $D=2$ case, a generic $D \geq 3$ model (1.1) complies with the pattern (1.9) as well.

In conclusion, we remark that the amplitude (4.8) exhibits purely topological assignement (see Section 6.2 for more details) of the $S(n_\phi)$-structures after the formal deformation $\{\Lambda_{n_\phi}^{(m_\phi)} \to (N^{n_\phi}\Omega_{n_\phi})^{m_\phi}\}$, $\{Q_{n_{\mu\nu}} \to 1\}$. In particular, consider the powers $m_\phi = \pm 1$ in which the relevant $S(n_\phi)$-twists Ω_{n_ϕ} enter the $\Lambda_{n_\phi}^{(m_\phi)}$ factors (4.1): $\{m_{\mu\nu} = 1\}$, $\{\, m_\rho = -1\}$, $m_+ = 1$. The latter are equal to the weights in the famous formula computing the Euler characteristic of a given cell-complex T: $2-2G_T = n_p - n_l + n_s = n_p m_p + n_l m_l + n_s m_s$, where n_p, n_l, n_s are the total numbers of respectively plaquettes, links, and sites of T. (The earlier heuristic arguments (in the case of the single chiral sector), consistent with the above pattern of the Ω_{n_ϕ} assignement, can be found in [22]).

The topological nature of the considered defomation of the amplitudes like (6.2) can be made transparent by the conjecture building on the $D = 2$ observation [18]. One may expect that, after this deformation, the TEK partition function $\tilde{X}_D$ yields the generating functional for the orbifold Euler characters of the Hurwitz-like spaces associated to the following set. The latter includes all the generalized branched covering spaces $\tilde{M}_\varphi$ (of the EK $2d$ cell-complex (2.2)) corresponding in eq. (6.2) to the considered deformation $\{P_{n_\phi}^{(N)} \to 1\}$. (For a close but somewhat distinct earlier conjecture, see [22].) Similar proposal can be made for the (weaker) 'topological' deformation $\{Q_{n_{\mu\nu}} \to 1\}$ of the PF $\tilde{X}_D$ reintroducing the $P_{n_\phi}^{(N)}$ factors.

6.1 The generalized covering spaces of $T_{EK} - \{q_s\}$.

Given the data in the argument of the δ_{n_+}-function in the l.h.s of eq. (6.2), our aim is to reconstruct the associated topological spaces $\tilde{M}_{EK} \equiv \tilde{M}_\varphi$ (i.e. the mappings (1.11)) which are summed over in the r.h.s. of (6.2). This problem, being *inverse* to what is usually considered in the framework of algebraic topology [26], will be resolved through the sequence of steps which appropriately generalizes the $D = 2$ analysis of the previous section. It is noteworthy that the spaces $\tilde{M}_{EK}$ do *not* coincide with the canonical branched covering spaces (BCSs) of the TEK $2d$ cell-complex T_{EK}. To say the least, a generic canonical BCS of T_{EK} is again a $2d$ cell-complex (with a number of branch points) but not a $2d$ surface like in the asserted mappings (1.11). As we will see, the basic amplitude (6.2) indeed complies with (1.11) and refers to the

novel class of the associated to T_{EK} spaces $\tilde{M}_{EK}$ to be called the generalized branched covering spaces (GBCSs) of T_{EK}.

Similarly to the $D = 2$ analysis, we start with the simpler case of the generalized covering spaces (GCSs) of T_{EK} corresponding to the deformation (6.3). Being specified by the immersions (1.11) without the (branch points' and collapsed subsurfaces') singularities, the GCSs can be reconstructed generalizing the $D = 2$ surgery-construction of the covering spaces. Take $D(D-1)/2$ $\mu\nu$-rectangulars $H_{\mu\nu}$ representing the 'decompactified' plaquettes of the EK base-lattice (2.2). Then, one is to begin with trivial ($n_{\mu\nu}$-sheet) coverings $\tilde{H}_{\mu\nu} = H_{\mu\nu} \otimes \Upsilon_{n_{\mu\nu}}$ of $H_{\mu\nu}$ with the edge paths $\otimes_q \alpha(\mu_q)\alpha(\nu_q)\beta^{-1}(\mu_q)\beta^{-1}(\nu_q)$, $q = 1, ..., n_{\mu\nu}$. To reproduce the effect of the σ_ρ-permutations in (6.2), first let us denote by $\{\alpha(\rho_k),\ k = 1, ..., n_\rho = \sum_{\nu\neq\rho}^{D-1} n_{\rho\nu}\}$ the ρ-set of the $\alpha(\rho_q)$ edges (collected from the $(D-1)$ different $\rho\nu$-plaquettes) ordered in accordance with the pattern (3.14) of the $S(n_\rho)$ basis $|I^{(\pm)}_{n(\rho)}>$. Similarly, we introduce the sets $\{\beta(\rho_k),\ k = 1, ..., n_\rho\}$.

To reconstruct the associated to (4.8) GCS of T_{EK}, at each particular ρ-link one is to perform the pairwise σ_ρ-identifications of $\alpha(\rho_k), \beta(\rho_k)$ according to the prescription (5.9). In this way, we have constructed appropriate conglomerate of the generalized Riemann surfaces $\tilde{M}(\{\sigma_\rho\})$ (with the total number D of the closed branch cuts) which are wrapped around T_{EK} in compliance with the mapping (1.11). Evaluating the Euler characteristic ε of $\tilde{M}(\{\sigma_\rho\})$ in accordance with (5.9), each connected component of the latter surface reveals the topology of a 2-tora. In turn, it matches with the overall power (6.4) of the $1/N$ factor which is inherited via (6.3) from the product (4.8) of the $\Lambda^{(m_\phi)}_{n_\phi}$ factors.

Given the GCSs $\tilde{M}(\{\sigma_\rho\})$ of T_{EK}, the branch points (encoded in (6.2) through the operators $\hat{T}_{\{p_{\mu\nu}\}}(\hat{T}_2^{(n_{\mu\nu})})^{i_{\mu\nu}}$) are reintroduced in essentially the same way as we did for the $D = 2$ case. The only subtlety is that, to make the employed cutting-gluing rules well-defined, it is convenient to view the EK cell-complex T_{EK} as the homotopy *retract* of a 'less singular' $2d$ complex T'_{EK} (possessing by construction the *same* fundamental group $\pi_1(T_{EK}) = \pi_1(T'_{EK})$). We refer the reader to Appendix C for the details, and now proceed with the $D \geq 3$ generalization of the $D = 2$ Gross-Taylor theorem concerning the interpretation of the symmetry factor entering the $D \geq 2$ stringy amplitudes like (1.9),(6.2).

6.2 The homomorphism of $\pi_1(T_{EK}-\{q_s\})$ into $S(n_+)$.

To effectively enumerate the constructed GBCSs of T_{EK} and their automorphisms, we first reconstruct what kind of homomorphism like (5.10) is encoded in the $D\geq 3$ pattern (6.2). It will provide with the precise mapping of the fundamental group $\pi_1(T_{EK}-\{q_s\})$ (of T_{EK} with a number of deleted points $\{q_s\}$ associated to the branch points) into the enveloping $S(n_+)$ group.

Observe first that the abstract group representation of $\pi_1(T_{EK})$ is defined, according to (2.2), in terms of the D generators α_ρ corresponding to the uncontractible cycles (i.e. the compactified ρ-links) of T_{EK}. Having excluded from T_{EK} a set of points $\{q_s\}$, it is convenient to recollect them into the three varieties of the ϕ-subsets $\{q_s\}=\cup_\phi\{q^{(\phi)}_{k_\phi}\}$, $\phi\in\{\mu\nu\},\{\rho\},+$, $k_\phi=1,...,b_\phi$, belonging respectively to the *interior* of the $\mu\nu$-plaquette, to the interior of the ρ-link, and to the single site of T_{EK}. The set of the $\pi_1(T_{EK}-\{q_s\})$ generators includes (additionally to the subset $\{\alpha_\rho\}$ inherited from $\pi_1(T_{EK})$) $\gamma^{(\phi)}_{k_\phi}$ associated to (the equivalence classes of) the closed paths encircling a single deleted point $q^{(\phi)}_{k_\phi}$. The representation of $\pi_1(T_{EK}-\{q_s\})$ is then completed by the $D(D-1)/2$ relations

$$\left([\alpha_\mu,\alpha_\nu]\quad\cdot\prod_{\phi=\mu\nu,\mu,\nu,+}\quad\prod_{k_\phi=1}^{b_\phi}\gamma^{(\phi)}_{k_\phi}\right)=1\quad;\qquad \mu\nu=1,...,D(D-1)/2, \tag{6.5}$$

each of which can be viewed as the $\mu\nu$-'copy' of (5.11).

Comparing the argument of the δ_{n_+}-function in the l.h.s. of eq. (6.2) with the pattern of (6.5), it is straightforward to write down the precise form of the relevant $D\geq 2$ homomorphism generalizing (5.10):

$$\psi:\ \psi(\alpha_\rho)=\sigma_\rho\ ;\quad \psi(\gamma^{(\phi)}_1)=\xi^{\{p_\phi\}}\ ;\quad \psi(\gamma^{(\mu\nu)}_{k_{\mu\nu}})=\tau^{(k_{\mu\nu}-1)}_{\mu\nu}\ ,\ k_{\mu\nu}\geq 2, \tag{6.6}$$

where $\xi^{\{p_\phi\}}\in T_{n_\phi}$, $\tau^{(k_{\mu\nu}-1)}_{\mu\nu}\in T_2^{(n_{\mu\nu})}$. Note that the three ϕ-varieties of the generators $\gamma^{(\phi)}_1$ match with the three species of $\hat{T}_{n_\phi}$ in eq. (6.2), while $\gamma^{(\mu\nu)}_{k_{\mu\nu}}$ represents the $k_{\mu\nu}$ th $\hat{T}_2^{(n_{\mu\nu})}$-factor in the product $(\hat{T}_2^{(n_{\mu\nu})})^{i_{\mu\nu}}$. In particular, eq. (6.6) implies that in eq. (6.5) $b_{\mu\nu}=i_{\mu\nu}+1$ while $b_+=b_\rho=1,\ \forall\rho$.

6.2.1 The symmetry factor.

To deduce the announced interpretation of $|C_\varphi(\{p_\phi\})|$ (entering the r.h.s. of eq. (6.2)), let us start with the following observation. The summation in the l.h.s of (6.2) can be viewed as the sum over the associated homomorphisms (6.6) constrained by the condition: $\xi^{\{p_\phi\}} \in T_{\{p_\phi\}}$ and $\tau_{\mu\nu}^{(k_{\mu\nu}-1)} \in T_2^{(n_{\mu\nu})}$. Therefore, one is to identify the proper equivalence classes of the homomorphisms (6.6) to parametrize uniquely the topologically inequivalent spaces $\tilde{M}_\varphi$ constructed in Section 6.1.

We assert that the two homomorphisms (6.6), ψ_1 and ψ_2, are *equivalent* (i.e. the corresponding generalized branched coverings $\tilde{M}_\varphi$ are homeomorphic) if and only if there exists some $\eta \in \otimes_{\mu\nu} S(n_{\mu\nu})$ so that

$$\psi_1(\zeta) = \eta\psi_2(\zeta)\eta^{-1} \quad , \quad \forall\zeta \in \pi_1(T'_{EK} - \{q_s\}) \quad ; \quad \eta \in \otimes_{\mu\nu} S(n_{\mu\nu}), \tag{6.7}$$

where the basis (3.32) for $\eta \in S(n_+)$ is implied. For a preliminary orientation, one observes that the conjugations (6.7) are induced by the $\otimes_{\mu\nu} S(n_{\mu\nu})$ permutations of the sheets (of $\tilde{M}_\varphi$) *separately* within each of the $D(D-1)/2$ distinct $n_{\mu\nu}$-subsets which covers (see Section 6.1) a given $\mu\nu$-tora $E_{\mu\nu}$ combined into T'_{EK}. (Complementary, $\otimes_{\mu\nu} S(n_{\mu\nu})$ is the *largest* subgroup of $S(n_+)$ providing with the conjugations leaving the argument of the δ_{n_+}-function in eq. (6.2) invariant.) We refer to Appendix D for the justification of the notion (6.7) of the equivalence and now simply deduce its consequences. Consider a particular branched covering $\tilde{M}_\varphi$, and let $C_\varphi(\{p_\phi\})$ is the group of the inequivalent automorphisms κ of $\tilde{M}_\varphi$: $\varphi \circ \kappa = \varphi$. Take the restriction of a given κ to the Υ_{n_+} space of $\{\varphi^{-1}(p)\}$). Akin to the $D = 2$ case, one shows that $C_\varphi(\{p_\phi\})$ is isomorphic to the conjugacy class (with respect to (6.7)) of the following subgroup of $\otimes_{\mu\nu} S(n_{\mu\nu})$. The latter is associated to such subset of conjugations (6.7) that leave all $\psi_1(\zeta)$ invariant: $\psi_1(\zeta) = \psi_2(\zeta)$, $\forall\zeta$, $\forall\eta(\kappa) \in C_\varphi(\{p_\phi\})$. In sum, there are exactly $(\otimes_{\mu\nu} n_{\mu\nu}!)/|C_\varphi(\{p_\phi\})|$ distinct homomorphisms $\{\psi_k\}$ associated to the same topology-type of $\tilde{M}_\varphi$ that justifies the basic identity (6.2).

7 The area-preserving homeomorphisms.

On a lattice, our $D \geq 3$ stringy proposal can be viewed as the confluence of the Wilson's string-like reformulation [6] of the (*finite* N) SC series with the power of the large N expansion. The feature, which sharply distinguishes the $D \geq 3$ Gauge String from the earlier $D \geq 3$ proposals [8, 7] in this direction, is the asserted invariance of the weights $w[\tilde{M}]$ under certain *continuous* group of the area-preserving worldsheet homeomorphisms. In $D \geq 3$, the latter symmetry is encoded in the following $D \geq 3$ 'descedant' inherited from the $D = 2$ renormgroup (RG) invariance of the YM_2 systems (1.1).

Recall first that a $D \geq 3$ YM_D theory (1.1) (having some lattice $\mathbf{L^D}$ as the base-space) can be equally viewed as the YM theory being defined on the $2d$ skeleton $\mathbf{T^D}$ of $\mathbf{L^D}$ represented by the associated $2d$ cell-complex. Consider the partition function (PF) of (1.1) on $\mathbf{T^D} = \cup_k E_k$ composed from the associated $2d$ surfaces E_k (of the areas A_k and with certain boundaries) according to the corresponding incidence-matrix [17]. Akin to the $D = 2$ case, this PF is invariant under subdivisions of $\mathbf{T^D}$ (preserving the total areas A_k) so that E_k can be made into the associated smooth $2d$ manifolds M_k (with boundaries) combined into a $2d$ cell-complex $\tilde{\mathbf{T}}^{\mathbf{D}} = \cup_k M_k$ (homeomorphic to $\mathbf{T^D}$). Therefore, refining the discretization, the YM_D lattice theory (1.1) can be adjusted to merge with the following continuous system. To implement the latter, we first put the associated to (1.1) continuous YM_2 theory (1.4) (keeping free boundary conditions) on each M_k . Then, one is to average over the gauge fields, 'living' on the boundaries of $\{M_k\}$, in compliance with the associated to $\tilde{\mathbf{T}}^{\mathbf{D}}$ *incidence-matrix*.

Similarly to the $D = 2$ case (see Section 1.1A), on the side of the proposed $D \geq 3$ Gauge String, the above relation to the continuous YM_2 system ensures the required invariance of the lattice string weights (entering the amplitudes defined by the $D \geq 3$ extension (1.11) of (1.9)). Indeed, on the one hand, both the parity P_φ and the symmetry factor $|C_\varphi|$ depend only on the *topology* of the worldsheet $\tilde{M}(T)$ and on the one of the corresponding taget-space $T = \cup_k E_k$. (Compare it with [8, 7] where the singular lattice *geometry* of T is 'built into' the curvature-defects.) On the other hand, the sum over the mappings φ also supports the required invariance of the set of relevant $\tilde{M}(T)$. Indeed, the multiple *integrals* (rather than discrete sums as in [8, 7]) over all admissible positions of the movable branch points (and

certain collapsed subsurfaces) span the *entire* interior of the discretized $2d$ surfaces E_k combined into T. The remaining 'discrete' contributions into $\int d\varphi$ refer to the purely topological assignement of the nonmovable singularities *anywhere* in the interiors of the appropriate subspaces of T. Altogether, the relevant group of the worldsheet homeomorphisms continuously translates the positions of the admissible singularities of the map (1.11) within the interiors of the associated subspaces of T.

To be more specific, we compare the weights $w[\tilde{M}]$ associated to the YM system (1.1) defined on a generic base-lattice F_{EK} homeomorphic to the 'elementary' EK $2d$ cell-complex (2.2). Let the total area (measured in the dimensionless units) of a given torus $E_{\mu\nu}$ is equal to $A_{\mu\nu}$. Combining the above general arguments with the analysis of Sections 3 and 4, the general weights on F_{EK} can be deduced from the basic T_{EK} amplitude (4.8) trading in each $Q_{n_{\mu\nu}}$-factor (4.5) the coupling constant λ for $\lambda A_{\mu\nu}$. In particular, the $(\Omega_{n_\phi})^{m_\phi}$-, σ_ρ-twist assignement is indeed purely topological: the corresponding conglomerates of the nonmovable branch points (and collapsed subsurfaces) can be placed anywhere (but without summation over positions) in the interior of the associated macroscopic ϕ-, ρ-cell of F_{EK}.

8 Smooth Gauge Strings vs. lattice ones.

It is clear that the method, we have developed in Sections 3-6 on the example of the TEK model (2.1), can be extended for the YM_D theories (1.1) defined on a generic regular subspace T of the $2d$ skeleton of the D-dimensional base-lattice $\mathbf{L^D}$. We defer the analysis of the generic weights (1.2) for a separate paper and now simply stress those implications of our present results which are T-independent and common for both the lattice and smooth realizations of the Gauge String. In this way, one actually decodes all the major peculiar features of the continuous theory of the smooth YM-fluxes which are novel compared to the conventional paradigm of the $D \geq 3$ 'fundamental' strings.

To begin with, one observes that the relevant smooth mapping (see e.g. eq. (1.11) for the worldsheets $\tilde{M}$ without boundaries) are allowed to have singularities which usually are not included into the $D \geq 3$ string-pattern (where the maps (1.11) are restricted to be sheer immersions). Complementary, certain class of the selfintersecting worldsheets $\tilde{M}$ is endowed in eq.

(1.2) with the extra factors $J[\tilde{M}(T)|\{\tilde{b}_k\}] \neq 1$ that does not have a direct counterpart in the known $D \geq 3$ string theories. In turn, it is the extra $J[..]$-weights which encode the data sufficient to reconstruct the associated continuous YM_D model (1.3)/(1.4). As for a single $D \geq 3$ Wilson loop average, the major novel ingredient is due to the various movable branch points and the collapsed $2d$ subsurfaces (assigned with the $\{\tilde{b}_k\}$-dependent weights (1.10)). Their positions can be viewed as the zero modes associated to the minimal surface contribution (provided the latter properly selfintersects) resulting in the extra area-dependence in the preexponent of $< W_C >$. This pattern generalizes the one of the $D = 2$ loop-averages [13, 7].

Next, let us demonstrate that the total contribution of the worldsheets $\tilde{M}$ (and, consequently, of the taget-spaces T) with the backtrackings vanishes which matches with the absence of foldings in the $D = 2$ Gross-Taylor representation (1.9). To make this feature manifest, we first recall that the relevant for (6.2) maps (1.11) (associated to the *SU(N)* TEK model) do not include any foldings of the worldsheets $\tilde{M}$ wrapped around T_{EK}. On the other hand, one could equally start with the large N *U(N)* TEK model where the proposed technique would result in all kinds of foldings 'covering' various conglomerates of the plaquettes (with possibly different $\mu\nu$-orientation). The comparison of these two complementary large N patterns justifies the above assertion.

On the side of the gauge theory defined on the standard cubic lattice $\mathbf{L^D}$, the TEK foldings are associated to the more general backtrackings of $\tilde{M}$. To see it, let us call worldsheets $\tilde{M}(T)$ (or taget-spaces T) without any backtrackings regular. Then, to any regular taget $T^{(r)}$ one can associate the variety of taget-spaces creating all kinds of the backtracking (bounding a zero 3-volume) with the support *not* necessarily belonging the original regular taget $T^{(r)}$ (in contradistinction to the foldings discussed in [7]).

The irrelevance of the backtrackings is intimately related to the invariance (in gauge theories) of the Wilson loop averages $< W_C >$ under the zig-zag backtrackings of the boundary contour C that is in sharp contrast with the situation in the conventional Nambu-Goto string and most of its existing generalizations. (Recently Polyakov [1] advocated the latter invariance as the crucial feature of the strings dual to gauge theories.) In the limit $N \to \infty$, the zig-zag symmetry of $< W_C >$ can be made manifest confronting as previously the two large N formulations of the open Gauge String: the

$U(N)$ one with a given backtracking contour C should be compared with the $SU(N)$ one with $\tilde{C}$ obtained contracting zig-zags of C.

8.1 Correspondence with the WC Feynman diagrams.

Let us reveal the WC/SC correspondence between the continuous YM_D models (1.4) in the WC phase and the associated smooth Gauge String in the SC regime. According to Section 7, once the backtrackings are absent, the relevant weights $w[\tilde{M}(T)]$ can be derived by putting the YM_2 system (1.4) onto a given $2d$ cell-complex (e.g. $2d$ surface) $T^{(r)} = \cup_k M_k$ where the $2d$ surfaces M_k are piecewise smooth. In the large N SC regime, the latter YM system is represented by the conglomerates of the worldsheets properly wrapped around $T^{(r)}$. On the other hand, in the large N WC regime the WC perturbation theory represents our system through the set the Feynman diagrams. Employing the standard path integral representation of the propagator of the free particle (in a curved space), the diagrams are visualized as the fishnet (of the gluonic trajectories) appropriately 'wrapped' around the same $2d$ complex $T^{(r)}$. The crucial observation is that, when $T^{(r)}$ is viewed as being embedded into $\mathbf{R^D}$, the latter fishnet can be reinterpreted as the specific contribution of the WC perturbation theory in the D-dimensional continuous YM_D model (1.4) (uniquely associated to the chosen YM_2 via eq. (1.3)). To select this contribution on the YM_D side, not only the gluonic trajectories in $\mathbf{R^D}$ should be constrained to have the space-time support on $T^{(r)}$ but also *each* colour a-component of the associated gluonic strength-tensor $F^a_{\mu\nu}(\mathbf{z})$, $\mathbf{z} \in T^{(r)}$, as the Lorentz (or $O(D)$) tensor should belong to the tangent space of $T^{(r)}$ at $\mathbf{z}$. (To circumvent gauge-fixing, tricky to explicitly match between the $T^{(r)}$- and standard formulations, one is to introduce an infinitesimally small mass term for the gauge field and then perform the above comparison.)

8.2 Suppression of the selfintersections.

At this step, it is appropriate to discuss a number of crucial simplifications inherent in the dynamics of the considered $D \geq 3$ continuous flux-theory compared to its lattice counterpart. Consider first the subset of smooth

closed $2d$ surfaces $\tilde{M}$ (resulting from the smooth *immersions* of a $2d$ manifold M into $\mathbf{R^D}$ which alternatively can be represented by the maps (1.11) with $T \in \mathbf{R^D}$) with selfintersections on submanifolds of the dimensionality $d > d_{cr} = (4 - D)$. The latter subset is of measure zero [23] in the set of all smooth *immersions* $M \to \mathbf{R^D}$ (with arbitrary selfintersections). Complementary, the smooth immersions $M \to \mathbf{R^D}$, $D \geq 4$, are dense [23] in the space of all *piecewise* smooth immersions $M \to \mathbf{R^D}$. In particular, in $D \geq 5$ the subset of φ-maps resulting in smooth embeddings (i.e. in closed non-selfintersecting $2d$ manifolds $\tilde{M}$) is the open, dense subspace of the space of all piecewise smooth immersions $M \to \mathbf{R^D}$. (Another example, in $D \geq 3$ the backtrackings (being viewed as two-dimensional selfintersections) are of measure zero, i.e. unstable.)

Next, suppose that the redefinition (discussed in the very end of Section 1 and after eq. (5.6)) of the bare $SU(N)$ string tension $\tilde{\sigma}_0$ is performed that eliminates certain types of the collapsed $2d$ subsurfaces originally built into (1.11). Then, by virtue of the Whitney immersion theorem [23], the singularities of the generic (piecewise smooth) mapping (1.11) in $D \geq 4$ can be restricted to the ones listed after eq. (1.11) with the exclusion (in the Heat-Kernal case (1.7)) of the 'movable' collapsed subsurfaces attached to a single sheet of the covering. As for the remaining admissible collapsed subsurfaces and the branch-points, being necessarily attached to nontrivial selfintersections of the worldsheet, they therefore correspond in $D \geq 5$ to 'measure-zero' *limiting points* of the dense subspace of the $2d$ manifolds $\tilde{M}$ (induced by the embeddings (1.11)) *without* boundaries. As for $D = 4$, the stable (i.e. of nonzero measure) selfintersections of a closed smooth surface $\tilde{M}$ can occur only at a set of *isolated* points. The advantage of the Gauge String is that, as it is clear from the previous sections, such zero-dimensional selfintersections are not weighted by any extra factors so that the corresponding worldsheets are still assigned with the $J[\tilde{M}(T)|\{\tilde{b}_k\}] = 1,\ b = 0,$ reduction of the weight (1.2). Altogether, it justifies the assertion made in the very end of Section 1.

To see how the $J[\tilde{M}(T)|\{\tilde{b}_k\}] \neq 1$ pattern is observable in $D \geq 3$, recall first the basic theorem [23] on the stability of the smooth immersions: the stability is equivalent to the local stability. In particular, it implies that the unstable in the case of a closed $2d$ surface selfintersections can not be stabilized (in the bulk) introducing some selfintersecting boundary contour(s). To

be more specific, consider the framework of the quasiclassical expansion for $< W_C >$ where the weight of the minimal surface enters as the isolated, discrete contribution disparate from the continuum of the string fluctuations. As a result of the above theorem, in $D \geq 4$ the only place where a nontrivial $J[\tilde{M}(T)|\{\tilde{b}_k\}] \neq 1$ factor may be observable (beyond the considered $SU(N)$ redefinition of $\tilde{\sigma}_0$) for a nonbacktracking contour C seems to be the contribution of the selfintersecting minimal surface. The simplest option, where the weights (1.10) of the movable branch points can be 'measured', is to take contours winding a number of times around a nonselfintersecting loop C. On the other hand, let C is any nonbacktracking loop associated to some nonselfintersecting 'minimal-area' surface(s). In this case, we expect that the simplest $J[\tilde{M}(T)|\{\tilde{b}_k\}] = 1,\ b = 1$ pattern of (1.2) (with the above redefinition of $\tilde{\sigma}_0$ and with the worldsheets represented by the smooth strict immersions $\tilde{M}:\ M \rightarrow \mathbf{R^D}$) in $D \geq 4$ is sufficient to reproduce the correct result for the contribution of the genus h smooth worldsheets to the average $< W_C >$.

9 Conclusions.

We propose the correspondence between the smooth Gauge String (1.2), induced from the lattice models (1.1), and the associated via (1.3) continuous YM_D theory (1.4) (with a finite UV cut off) in the large N SC phase. This duality implies the concrete prediction (1.6) for the bare string tension $\sigma_0 = \Lambda^2 \tilde{\sigma}_0$ as the function of the coupling constants entering the YM_D lagrangian (1.4). In particular, it readily allows for a number of nontrivial large N predictions in the extreme SC limit where σ_0 merges with the leading asymptotics of the physical string tension σ_{ph}. More generally, the correspondence asserts that the generic continuous YM_D model (1.4) is confining in $D \geq 3$ at least in the large N SC regime accessible by our approach.

In turn, it suggests the mechanism of confinement in the standard weakly coupled (WC) $D = 4$ continuous gauge theory (1.8) at large N. Consider the effect of the *Wilsonian* renormgroup flow on the original YM_D theory in the WC phase at the UV scale (where the large N SC expansion, in terms of the proposed microscopic gauge strings, fails). The idea is that the latter YM_D theory, at the sufficiently low-energy scale, may be superseded by such

effective strongly coupled YM_D system where the (effective) Gauge String representation is already valid. As the effective YM_D system is *quasilocal*, we assert that in the effective Gauge String the Nambu-Goto term in (1.2) is traded for the whole operator expansion (OPE) running in terms of the extrinsic and intrinsic curvatures of the worldsheet. (Complementary, the weights of the movable branch points are modified in such a way that the integration, over the positions of the latter points, results in exactly the same pattern of the worldsheet OPE as for the descendant of the Nambu-Goto term.)

Finally, let us make contact with the two alternative stringy proposals [5],[1]. As for [5], Witten argues that in continuous YM_4 theory (with a finite UV cut off Λ) the physical string tension σ_{ph} in the extreme large N SC limit $g^2N \to \infty$, $g^2 \sim O(1/N)$, scales as $\sigma_{ph} \sim g^2 N \Lambda^2$. On the other hand, eq. (1.6) predicts similar large N SC $O(N\tilde{g}^2)$-scaling of the bare string tension σ_0 (where $\tilde{g}^2 \equiv \tilde{g}^2(\{\tilde{b}_k\}, N) \sim O(N^{-1}\Lambda^0)$) in the following subclass of the YM_D systems. The latter are defined by the subset of the actions (1.4) with the coefficients g_r constrained by

$$\tilde{g}^2 N \to \infty \ : \ [g_r]^{-1} \sim O(\Lambda^{-2n+4}[N^{\frac{1}{2}}\tilde{g}]^{-n+\gamma(r)} N^{2-\sum_{k=1}^{n} p_k}) \ , \qquad (9.1)$$

where $\gamma(r) \geq 0$ (and there is at least one irrep r for which $\gamma(r) = 0$). According to the structure of the (abelianized) Born-Infeld action, the Witten's prescription [5] is supposed to induce the YM_D action with the local part belonging to the variety (9.1) (with possible addition of the commutator-terms which does not alter the conclusions). In sum, our prediction (1.6) for the string tension in the extreme large N SC limit semiquantitatively matches with the $D = 4$ pattern of [5] motivated by the AdS/CFT correspondence.

As for the Polyakov's $D = 4$ Ansatz [1], it is aimed at a stringy reformulation of the continuous YM_4 theory (1.8) in the large N WC regime. Assuming that the general pattern of the Ansatz is applicable to the SC phase as well, an indirect comparison might be possible. In particular, the advocated by Polyakov invariance of the worldsheet action under the extended group of the diffeomorphisms (with the singularities due to the zero Jacobian) matches with the two key features of the Gauge String: the vanishing contributions of the worldsheets $\tilde{M}$ with the backtrackings and the invariance of $w[\tilde{M}]$ under the group of the area-preserving diffeomorphisms specified in the Section 7.

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A: Reconstruction of $\Xi_{4n_+}(\{R_{\mu\nu}\})$.

To derive the $S(4n_+)$ operator $\Xi_{4n_+}(\{R_{\mu\nu}\})$ of eq. (3.3), we first rewrite the characters of the original expression (2.5) in terms of certain $S(4n_{\mu\nu})$ operator $\mathbf{D}(\Xi_{4n_{\mu\nu}}(R_{\mu\nu}))$

$$\chi_{R_{\mu\nu}}(U_\mu U_\nu U_\mu^+ U_\nu^+) = Tr_{n_{\mu\nu}}[\ \mathbf{D}(\Xi_{4n_{\mu\nu}}(R_{\mu\nu}))\ \mathbf{D}_2(\{U_\rho \otimes U_\rho^+\})\ ]\ , \qquad \text{(A.1)}$$

where $\mathbf{D}_2(\{U_\rho \otimes U_\rho^+\})$ is the $D=2$ option of (3.4). In the $|\tilde{I}_{4n(\mu\nu)}>$-basis (3.22), $\mathbf{D}(\Xi_{4n_{\mu\nu}}(R_{\mu\nu}))$ reads explicitly

$$\sum_{\sigma\in S(n_{\mu\nu})} \frac{\chi_{R_{\mu\nu}}(\sigma)}{n_{\mu\nu}!} \delta^{p_1}_{l_{\sigma(3n+1)}}..\delta^{p_n}_{l_{\sigma(4n)}}\ \delta^{p_{n+1}}_{l_1}..\delta^{p_{2n}}_{l_n}\ \delta^{p_{2n+1}}_{l_{n+1}}..\delta^{p_{3n}}_{l_{2n}}\ \delta^{p_{3n+1}}_{l_{2n+1}}..\delta^{p_{4n}}_{l_{3n}} \qquad \text{(A.2)}$$

which can be represented in the concise form of the *inner*-product

$$\Xi_{4n_{\mu\nu}}(R_{\mu\nu}) = \Psi_{4n_{\mu\nu}} \cdot \tilde{P}_{4n_{\mu\nu}}(R_{\mu\nu}) \quad ; \quad \tilde{P}_{4n}(R) = \hat{1}^{\oplus 3}_{[n]} \otimes C_R\ , \qquad \text{(A.3)}$$

where $\Psi_{4n_{\mu\nu}}$ is defined by eq. (3.20). Each of the four (ordered) $S(n_{\mu\nu})$-operators in the outer product composition of $\tilde{P}_{4n_{\mu\nu}}(R_{\mu\nu})$ are postulated to act on the corresponding four (ordered) $S(n_{\mu\nu})$-subspaces (3.22) of the space $|\tilde{I}_{4n(\mu\nu)}>$. Finally, in the $S(4n_+)$ basis (3.17), one evidently obtains $\Xi_{4n_+}(\{R_{\mu\nu}\}) = \otimes_{\{\mu\nu\}}\Xi_{4n_{\mu\nu}}(R_{\mu\nu})$ that matches with eq. (3.19) modulo a slight deviation of $\tilde{P}_{4n_{\mu\nu}}(R_{\mu\nu})$ from $P_{4n_{\mu\nu}}(R_{\mu\nu})$ (defined by eq. (3.23)).

The possibility to substitute in eq. (3.3) the operator $\tilde{P}_{4n_{\mu\nu}}$ by $P_{4n_{\mu\nu}}$ is ensured by the following feature of the pattern (A.1). Owing to the basic commutativity $[\mathbf{D}(\sigma), U^{\oplus n}] = 0\,, \quad \forall\sigma \in S(n)$, any of the unity operators $\hat{1}_{[n]}$ (the operator $\tilde{P}_{4n}$ is composed of) can be substituted by the C_R-factor properly weighted according to the multiplication rule $(C_R)^2 = C_R/d_R$. Remark also that in eq. (A.3) the relative order of the factors $\Psi_{4n_{\mu\nu}}$ and $\tilde{P}_{4n_{\mu\nu}}$ is immaterial since $[\ \Psi_{4n}\ , \otimes_{k=1}^4 \sigma_k\ ] = 0,\ \forall\sigma_k \in S(n)$.

Next, let us prove that, in the dual representation (3.24) of (2.5), the substitution (3.27) (of $\otimes_{\rho=1}^{D}\Phi_{2n_\rho}$ by its square) doesn't change the character (3.5). The latter 'symmetry' can be traced back to the fact that the multiple integral (2.5), represented by the $S(4n_+)$ element (3.24), is a real-valued function. To take advantage of this fact, observe that $\tilde{\Phi}_{2m} = (\Phi_{2n}\otimes\Phi_{2n})$, $m = 2n$, is the operator which represents the complex conjugation of the characters $\chi_R(U_{\mu\nu}) = Tr_{4n}[\mathbf{D}(\Xi_{4n})\tilde{U}_{\mu\nu}^{\oplus n}]$ entering (2.5):

$$Tr_{4n_{\mu\nu}}[\mathbf{D}(\Xi_{4n_{\mu\nu}})\,\tilde{U}_{\mu\nu}^{\oplus n_{\mu\nu}}]^* = Tr_{4n_{\mu\nu}}[\mathbf{D}(\Xi_{4n_{\mu\nu}}\tilde{\Phi}_{2m_{\mu\nu}})\,\tilde{U}_{\mu\nu}^{\oplus n_{\mu\nu}}], \tag{A.4}$$

where $\Xi_{4n_{\mu\nu}} \equiv \Xi_{4n_{\mu\nu}}(R_{\mu\nu})$ is defined in eq. (A.3), and $\tilde{U}_{\mu\nu}$ is introduced in eq. (3.28). As the master-integral (2.5) is invariant under the simultaneous transformation (A.4) of all the involved characters while $\otimes_{\mu\nu}\tilde{\Phi}_{2m_{\mu\nu}} = \otimes_{\rho=1}^{D}\Phi_{2n_\rho}$, we arive at the required invariance under (3.27).

As for eq. (A.4), it can be deduced by linearity from the following more elementary identity. To formulate the latter, let us first introduce the representation of the basic traces (the characters (A.4) are composed of)

$$tr((U_\mu U_\nu U_\mu^+ U_\nu^+)^n) \equiv Tr_{4n}[\mathbf{D}(\Gamma_{4n}\Psi_{4n})\tilde{U}_{\mu\nu}^{\oplus n}] = Tr_{4n}[\mathbf{D}(\Gamma_{4n}^{-1}\Psi_{4n})\tilde{U}_{\mu\nu}^{\oplus n}], \tag{A.5}$$

where $\Gamma_{4n} = (c_n \otimes \hat{1}_{[n]}^{\oplus 3})$ in the $S(4n)$ basis (3.22), and c_n is the n-cycle permutation. Then, the required identity reads (with $m = 2n$)

$$Tr_{4n}[\mathbf{D}(\Gamma_{4n}\ \Psi_{4n})\ \tilde{U}_{\mu\nu}^{\oplus n}]^* = Tr_{4n}[\mathbf{D}(\Gamma_{4n}\ \Psi_{4n}\ \tilde{\Phi}_{2m})\ \tilde{U}_{\mu\nu}^{\oplus n}]. \tag{A.6}$$

For its justification. we first introduce the tensor representation of the relevant complex conjugation: $Tr_n[\mathbf{D}(\sigma)\otimes_{k=1}^{n} U_k]^* = Tr_n[\mathbf{D}(\sigma^{-1})\otimes_{k=1}^{n} U_k^+]$ where the orderings of the U_k factors in the l.h.s. is the same as that of U_k^+ in the r.h.side. The key-observation is that, when $\otimes_{k=1}^{n} U_k^+ = \tilde{U}_{\mu\nu}^{\oplus n_{\mu\nu}}$, the substitution $U_k \to U_k^+$ can be performed as the conjugation with respect to $(\Phi_{2n_{\mu\nu}}\otimes\Phi_{2n_{\mu\nu}}) = \tilde{\Phi}_{2m_{\mu\nu}} = \tilde{\Phi}_{2m_{\mu\nu}}^{-1}$: $\tilde{\Phi}_{2m_{\mu\nu}}^{-1}\cdot\tilde{U}_{\mu\nu}^{\oplus n_{\mu\nu}}\cdot\tilde{\Phi}_{2m_{\mu\nu}} = \tilde{V}_{\mu\nu}^{\oplus n_{\mu\nu}}$ where $\tilde{V}_{\mu\nu} = U_\mu^+\otimes U_\mu\otimes U_\nu^+\otimes U_\nu$. This is because, owing to eq. (3.16), Φ_{2n} interchanges (either upper or lower) indices between the $U^{\oplus n}$ and $(U^+)^{\oplus n}$ blocks of $U^{\oplus n}\otimes(U^+)^{\oplus n}$. As in eq. (A.5) Γ_{4n} can be substituted by Γ_{4n}^{-1} (while Γ_{4n} commutes with both $\tilde{\Phi}_{2m}$ and Ψ_{4n}), all what remains to be proved is that $\Psi_{4n}\tilde{\Phi}_{2m} = \tilde{\Phi}_{2m}\Psi_{4n} = \Psi_{4n}^{-1}$, $n = 2m$. The last identity, owing to the specific patterns (3.16) and (3.20) of Φ_{2n} and Ψ_{4n}, directly follows from its reduced $m = 2$ variant. This completes the justification of (A.6).

Finally, let us sketch the proof of the basis formula (3.31). To make sure that the contraction of the $\sigma_\rho^{(\pm)}$ indices is the same in the both sides of (3.31), the following decomposition of the index-structure is helpful. Building on the prescription (3.6), we rewrite the $S(n_\rho)$ operators $D(\sigma_\rho^{(\pm)})$ in the form

$$D(\sigma_\rho^{(\pm)})^{\{j^{\oplus n_\rho}\}}_{\{i^{\oplus n_\rho}\}} \equiv D(\sigma_\rho^{(\pm)})^{\{j^{\oplus n_{\rho\mu}}\}...\{j^{\oplus n_{\rho\nu}}\}}_{\{i^{\oplus n_{\rho\mu}}\}...\{i^{\oplus n_{\rho\nu}}\}} \quad , \quad \mu, ..., \nu \neq \rho, \tag{A.7}$$

where each of the $D-1$ $n_{\rho\lambda}$-subsets of the indices ($\lambda \neq \rho$) acts on the associated $S(n_{\rho\lambda})$ subspace (3.32) of the enveloping space $S(n_+)$. Let consider any $n_{\rho\nu}$-subset, associated to $\sigma_\rho^{(\pm)}$, as the composite block-index $j^{(\pm)}_{\rho\nu}$. A direct inspection then reveals that, in *both* sides of eq. (3.31), for a given μ, ν the four block-indices $j^{(\pm)}_{\mu\nu}, j^{(\pm)}_{\nu\mu}$ (corresponding to either $\rho = \mu$ or $\rho = \nu$) are c_4-cyclically contracted according to the pattern (3.19)/(3.20) of $\otimes_{\mu\nu}\Psi_{\mu\nu}$.

As for the ordering of the inner ρ-products in the r.h.s. of (3.31), it should be deduced from the particular ordering of the $|i_\pm(\rho)>$-blocks composed into the elementary subspace (3.21) used to define the $D(D-1)/2$ operators $\Psi_{4n_{\mu\nu}}$. To justify the prescription stated in Section 3, one is to combine the above analysis with the specified pattern of the ordering in the elementary $D=2$ case (3.29).

B: Tensor representation vs. Regular one.

Let us first derive the identity (2.8). Actually, this equation is nothing but the transformation of the trace of the tensor $\mathbf{D}(\sigma)$-representation (2.7) into that of the canonical regular representation X_{REG} [16] (both associated to a given $S(n)$-algebra). Recall that X_{REG} is defined [16] on the vector space $\Theta = \{\sigma_i;\ \sigma_i \in S(n)\}$ by the homomorphism $S(n) \to M_i \in End(\Theta)$ of the $S(n)$-group into the group of the endomorphisms of Θ: $\sigma_i\sigma_j = \sigma_k\Delta^k_{ij} \to (M(\sigma_i))^k_j \equiv (M_i)^k_j = \Delta^k_{ij}$. Here $\sigma_i\sigma_j = \sigma_q$, and $\Delta^k_{ij} = 1$ or 0 depending on whether $q = k$ or $q \neq k$. Defined in this way, the matrices M_i satisfy the same relations $M_iM_j = M_q$ as the origibal $S(n)$ group-elements σ_i.

To make contact with the tensor representation (2.7), recall [16] first that

$$\chi_{REG}(\sigma) = \sum_{R\in Y_n} d_R\ \chi_R(\sigma) = n!\ \delta_n(\sigma) \quad , \quad \chi_{REG}(P_R\sigma) = d_R\chi_R(\sigma), \tag{B.1}$$

where $tr[M(\sigma)] \equiv \chi_{REG}(\sigma)$ and $P_R = d_R C_R$. As for $\delta_n(..)$, on the $S(n)$-group it reduces to the standard Kronecker δ-function: $\delta_n(\sigma) = \delta[\sigma, \hat{1}_{[n]}]$ (with $\hat{1}_{[n]}$ being the 'trivial' unity-permutation of $S(n)$). By linearity, it is then extended to the $S(n)$-algebra. Finally, rewriting the $V = \hat{1}$ reduction of the second Frobenius formula (see e.g. [18, 25]) in terms of $\chi_{REG}(C_R\sigma)$

$$Tr_n[\mathbf{D}(\sigma)] = \sum_{R\in Y_n^{(N)}} \chi_R(\sigma)\chi_R(V)|_{V=\hat{1}} = \sum_{R\in Y_n^{(N)}} dimR\ \chi_{REG}(C_R\sigma), \tag{B.2}$$

and employing the definition (3.9) of $\Lambda_n^{(1)}$ (where the sum runs over the chiral *U(N)* irreps *R*), we arrive at the basic identity (2.8). (Omitting the $P_n^{(N)}$ projector, the latter identity was discussed in [22].)

In conclusion, let us briefly sketch the derivation of eqs. (4.1), (4.4). As for (4.1), in its l.h.s. we use first (4.3) to substitute $d_R(n!dimR/d_R)^m$ by $\chi_R((N^n\Omega_n)^m)$. Replacing $C_R = \sum_{\sigma\in S(n)} \chi_R(\sigma^{-1})\sigma/n!$, we then combine the two resulting characters into one

$$\frac{1}{n!}\sum_{\sigma\in S(n)}\sum_{R\in Y_n} d_R\ \chi_R((N^n\Omega_n)^m P_n^{(N)}\sigma^{-1})\ \mathbf{D}(\sigma)\ . \tag{B.3}$$

As $[\Omega_n, \sigma] = 0,\ \forall\sigma \in S(n)$, for the derivation of (B.3) we used the identity $\chi_R(\Psi\sigma) = \chi_R(\sigma)\chi_R(\Psi)/d_R,\ \forall\rho \in S(n)$, if $[\Psi, \rho] = 0$ (which directly follows from the possibility [16] to expand any such Ψ, in the center of $S(n)$, in terms of the Young projectors $\Psi = \sum_{R\in Y_n} \psi_R P_R$). We have used also that $\chi_R(P_n^{(N)}\sigma) = \chi_R(\sigma)$ or 0 depending on whether or not $R \in Y_n^{(N)}$. Applying the completeness condition $\sum_{\sigma\in S(n)} \delta_n(\sigma^{-1}\Phi)\ \mathbf{D}(\sigma) = \mathbf{D}(\Phi)$ (where Φ is *any* element of the $S(n)$-algebra) we arrive at the announced result (4.1).

Concerning eq. (4.4), first one is to represent (see e.g. [13]): $C_2(R) = (nN + 2\chi_R(\hat{T}_2^{(n)})/d_r - n^2/N),\ R \in Y_n,$ which is a particular case of the general relation (E.1). Expanding $\chi_R(\hat{T}_2^{(n)})/d_r$ (where $\hat{T}_2^{(n)}$ is defined by eq. (4.5)) into the preexponent, the derivation of (4.4) is given by a minor modification of the steps developed in the context of (4.1).

C: Reintroducing the branch points.

Given the generalized covering spaces (GCSs) $\tilde{M}(\{\sigma_\rho\})$ of T_{EK} (resulting after the deformation (6.3) of (4.8)), the generalized branched covering spaces

(GBCSs) of T_{EK} (encoded in the full expression (6.2)) can be reconstructed reintroducing onto $\tilde{M}(\{\sigma_\rho\})$ the relevant branch points (BPs). Alternatively, the GBCSs of T_{EK} are reinterpreted as the GCSs of $T_{EK} - \{q_s\}$ where the deleted (from T_{EK}) set of points $\{q_s\}$ is associated to the corresponding BP permutations $\xi^{\{p_\phi\}}$, $\tau_{\mu\nu}^{(k_{\mu\nu})}$ in compliance with the homomorphism (6.6). Then, a particular GBCS of T_{EK} is reproduced from the associated GBC (of $T_{EK} - \{q_s\}$) 'closing' the temporarily deleted points $\{\varphi^{-1}(q_s)\}$. At a given q_s, the closure [15, 18] is performed as the mapping of the $n_{\phi(s)}$-set $\varphi^{-1}(q_s) \cong \Upsilon_{\phi(s)}$ onto set of points matching with the number of cycles in the cyclic decomposition of the associated to q_s permutation (6.6).

To make the above procedure well-defined, I propose to view T_{EK} as the *homotopy retract* of a 'less singular' $2d$ cell-complex T'_{EK} with the same fundamental group: $\pi_1(T'_{EK}) = \pi_1(T_{EK})$. As the proposed retraction preserves the basic homomorphism (6.6), the GBCSs of T_{EK} can be consistently treated as the limiting case of the corresponding GBCSs of T'_{EK}.

First, let us thicken each ρ-link into an infinitesimally thing cylinder $\bar{Z}_\rho = \cap_{\nu\neq\rho} E_{\rho\nu}$ shared by the $D-1$ corresponding 2-tora $E_{\rho\nu}$. (To match with the canonical construction [15, 18] of the branch points, (for a given ρ) all $E_{\rho\nu}$, $\nu \neq \rho$, can be always adjusted to have the *same* orientation when restricted to $\bar{Z}_\rho$.) Complementary, the intersection $\bar{Z}_+ = \cap_\rho \bar{Z}_\rho = \cup_{\rho\nu} E_{\rho\nu}$ of the D different cylinders $\bar{Z}_\rho$ (or, equivalently, of all the $D(D-1)/2$ $\mu\nu$-tora $E_{\mu\nu}$) is thicken into an infinitesimal disc $\bar{Z}_+$. (Again, all the cylinders $\bar{Z}_\rho$ can be adjusted to have the *same* orientation when restricted to $\bar{Z}_+$.)

Choose a base-point $p \neq q_s$, $\forall s$, (common for all the equivalence classes of the pathes represented in the constraint (6.5)) in the interior Z_+ of $\bar{Z}_+$. We require that the location of the set $\{q_s\} = \cup_\phi \{q_{k_\phi}^{(\phi)}\}$ complies with

$$q_{k_{\mu\nu}}^{(\mu\nu)} \in (E_{\mu\nu} - (\bar{Z}_\mu \cup \bar{Z}_\nu)) \;\; ; \;\; q_{k_\rho}^{(\rho)} \in (Z_\rho - \bar{Z}_+) \;\; ; \;\; q_{k_+}^{(+)} \in Z_+ \; , \tag{C.1}$$

where Z_ϕ is the *interior* of the closed space $\bar{Z}_\phi$, $\phi \in \{\mu\nu\}, \{\rho\}, +$. Also, the introduced in Section 6 closed branch cuts ϖ_ρ (inherited from the GCS of T'_{EK}) are supposed to satisfy: $\varpi_\rho \in Z_\rho$, while $\{q_{k_\phi}^{(\phi)}\} \cap \varpi_\rho = 0$, $\forall \rho$, $\forall k_\phi$.

To reintroduce the branch points onto the GCSs $\tilde{M}(\{\sigma_\rho\})$ of T'_{EK}, one is to consider the additional (to ϖ_ρ) branch cuts $\varpi_{\phi(s)}$ outgoing from the corresponding points q_s of T'_{EK}. (Combining all the cuts together, one is

supposed to arrive at a network $\Omega = \{\varpi_{\phi(s)}, \varpi_\rho\}$ whose global consistency is ensured by the δ_{n_+}-constraint (6.2).) To implement $\varpi_{\phi(s)}$, we cut the GCS $\tilde{M}(\{\sigma_\rho\})$ along the φ^{-1}-image of $\varpi_{\phi(s)}$ (starting at q_s and terminating either at some other point q_k or, possibly, at an auxiliary vertex of the network Ω). Denote $\alpha(s_k)$ and $\beta(s_k)$ (with $k = 1, ..., n_{\phi(s)}$) the resulting $n_{\phi(s)}$ boundaries of the sheets (of $\tilde{M}(\{\sigma_\rho\})$) respectively on the left and on the right sides of $\varpi_{\phi(s)}$. To obtain the GCS of $T'_{EK} - \{q_s\}$ corresponding to the homomorphism (6.6), one is to perform the pairwise identifications of these boundaries according to the developed prescription (5.9). Matching with (6.2), one simply substitutes in (5.9): $\rho \to s$. so that σ_s is either $\xi^{\{p_\phi\}}$ or $\tau_{\mu\nu}^{(k_{\mu\nu})}$ depending on the image of the homomorphism (6.6) assigned to a given branch point q_s constrained by (C.1).

D: Counting the generalized coverings.

The proof, that inequivalent spaces $\tilde{M}_\varphi$ in (6.2) are parametrized by the equivalence classes (6.7) of the homomorphisms (6.6), appropriately generalizes the analogous proof [26, 15] for the canonical coverings employed in [13, 18]. First, one is to demonstrate that the relevant inequivalent spaces $\tilde{M}_\varphi$ entering (6.2) are uniquely parametrized by certain $\otimes_{\mu\nu} S(n_{\mu\nu})$ conjugacy classes of the $\pi_1(T'_{EK} - \{q_s\}|p)$ subgroups. Then, one proves the one-to-one correspondence between the latter classes of the subgroups and the equivalence classes (6.7) of the homomorphisms (6.6). Let us outline the above proofs with the emphasis on the subtleties novel compared to [26, 15].

To begin with, we observe that (by the same token [26] as in the canonical case) any subgroup of $\pi_1(T'_{EK} - \{q_s\}|p)$ can be viewed as the image

$$\varphi_*(\pi_1(\tilde{M}_\varphi - \{\varphi^{-1}(q_s)\}|\tilde{p})) \quad , \quad \tilde{p} \in \varphi^{-1}(p) \cong \Upsilon_{n_+} \; , \tag{D.1}$$

induced by the φ-mapping (1.11): $\tilde{M}_\varphi \to T'_{EK}$, reconstructed in Section 6.1 and Appendix C. The consistency of (D.1) implies that a given $\tilde{M}_\varphi$ yields some branched covering (with n_+-sheets) 'above' the single site s of T_{EK} which is 'regularized', $s \to Z_+$ (see eq. (C.1) of Appendix C), into an infinitesimal disc Z_+ of T'_{EK} so that $p \in Z_+$. (In contradistinction to the canonical case, the point p is *not* allowed to leave Z_+.) Perform a

shift of the base-point $\tilde{p} \in \varphi^{-1}(p)$, located on a jth sheet of the branched covering of Z_+, to $\tilde{p}' \in \varphi^{-1}(p)$ on some other kth sheet along the path $\varepsilon \in \tilde{M}_\varphi - \{\varphi^{-1}(q_s)\}$. It induces [26] the associated conjugation of the $\pi_1(T'_{EK} - \{q_s\}|p)$ elements (and, hence, its subgroups) with respect to the element given by the image $\varphi(\varepsilon) \in \pi_1(T'_{EK} - \{q_s\}|p)$ of the path ε.

Next, one notes that (similarly to the canonical construction [26]) the shift ε acts on the n_+-set $\varphi^{-1}(p) \cong \Upsilon_{n_+}$ as the simple $S(n_+)$-transposition $\psi(\varphi(\varepsilon)) \in T_2^{(n_+)}$ of the two corresponding entries: $j \to k,\ j,k,= 1,...,n_+$. As a result, ε induces the $S(n_+)$ conjugation (6.7) of (6.6) with $\eta = \psi(\varphi(\varepsilon))$ that can be visualized as the permutation of the two associated sheets of $\tilde{M}_\varphi$. The latter does not necessarily leaves intact the topology of $\tilde{M}_\varphi$. The cutting-gluing technique of Section 6.1 implies that interchanges of the sheets encoded in the conjugations with $\eta \in S(n_+)/\otimes_{\mu\nu} S(n_{\mu\nu})$ (i.e. between the $n_{\mu\nu}$-sheet coverings of different $\mu\nu$-tora $E_{\mu\nu}$ of T'_{EK}) should be excluded when collecting an equivalence class of maps (6.7) corresponding to a given topology of $\tilde{M}_\varphi$.

Consider the $\otimes_{\mu\nu} S(n_{\mu\nu})$ conjugacy classes of the $\pi_1(T'_{EK} - \{q_s\}|p)$ subgroups induced by the combination of the elements $\varphi_*(\varepsilon)$ which are the images of those of the shifts ε that connect the sheets within the $n_{\mu\nu}$-sheet covering of a given torus $E_{\mu\nu}$ of T'_{EK}. Taking into account the above discussion, one readily modifies the canonical construction [26] to prove that the considered classes of the subgroups are indeed in the 1-to-1 correspondence with the inequivalent generalized covering spaces $\tilde{M}_\varphi - \{\varphi^{-1}(q_s)\}$ of $T'_{EK} - \{q_s\}$ (and, therefore, with inequivalent GBCSs $\tilde{M}_\varphi$ of T'_{EK}). In particular, the cutting-gluing rules of Section 6.1 ensure that the mapping from $\tilde{M}_\varphi$ to the corresponding classes of the $\pi_1(T'_{EK} - \{q_s\}|p)$ subgroups is onto.

Finally, let us reveal the second 1-to-1 correspondence of the former classes of the subgroups with the equivalence classes (6.7) of the homomorphisms (6.6). From above, it is clear that the ψ-images (6.6) of these $\pi_1(T'_{EK} - \{q_s\}|p)$ subgroups can be related by the $\otimes_{\mu\nu} S(n_{\mu\nu})$ permutations corresponding to the interchanges of the sheets separately within each of the $D(D-1)/2$ $n_{\mu\nu}$-sheet coverings associated to particular $E_{\mu\nu}$-tora of T'_{EK}. As a result, akin to the $D = 2$ case [26, 15], the latter permutations are represented by the required conjugations (6.7) with $\eta \in \otimes_{\mu\nu} S(n_{\mu\nu})$. Conversely, equivalent homomorphisms determine equivalent $\otimes_{\mu\nu} S(n_{\mu\nu})$ classes of the

$\pi_1(T'_{EK} - \{q_s\}|p)$ subgroups (according to the discussed above construction of $\eta = \psi(\varphi(\varepsilon))$). Summarizing, it completes the proof of the two asserted correspondences.

E: The higher Casimirs' actions.

To derive the generalized form (4.6),(4.7) of $Q_n(\Gamma)$ in (4.4), first one is to begin with the (Schur-Weyl) duality [18, 14] between the Casimir eigenvalues

$$\frac{C_q(R)}{N^{q-1}} = \sum_{T_{\{p\}} \in S(n)} a_q(N, n, T_{\{p\}}) \ \frac{\chi_R(\hat{T}_{\{p\}})}{d_R} \quad , \quad R \in Y_n^{(N)} \ , \tag{E.1}$$

and the symmetric group characters $\{\chi_R(\hat{T}_{\{p\}})\}$ (where $\hat{T}^{(n)}_{\{p\}} \equiv \hat{T}_{\{p\}}$ is defined by eq. (4.6), while $1 \le q \le N$). The coefficients $a_q(...)$ are defined by the formular [18, 14] which can be deduced from eq. (4.7) via the substitution: $\upsilon_{\{p\}}(.., n, N) \ \rightarrow \ a_q(N, n, T_{\{p\}})$, $s_{\{p\}}(\{\tilde{b}_r\}, m, l) \ \rightarrow \ s_q(T_{\{p\}}, m, l)$, $M_{\{p\}} \ \rightarrow \ M_0(T_{\{p\}})$. In particular, (keeping $n \sim O(N^0)$) the *leading* term of the formal $1/N$ expansion of the pth order Casimir operator reads: $C_q(R) = N^{q-1}(n + O(N^{-1}))$, $R \in Y_n^{(N)}$, which corresponds in (E.1) to the trivial unity permutation $T_{\{p\}} = \hat{1}_{[n]}$. As for the remaining $T_{\{p\}} \neq \hat{1}_{[n]}$ contributions, the branch point interpretation of the $l = 0$ term in expression like (4.7) is discussed in the end of Section 5 (in fact $s_q(T_{\{p\}}, m, 0) = 0$ for $m \ge 2$). To interprete a given $l \ge 1$ term, first one is to resum the powers n^m in terms of the numbers of inequivalent subdivisions of n objects into two subsets containing t and $(n-t)$ objects: $n^m = \sum_{t=1}^m f_{m,n} n!/t!(n-t)!$. As a result, similarly to [14, 18] the latter contributions include (in addition to the $T_{\{p\}}$ branch point) the extra collapsed to a point $2d$ subsurfaces. Being cut out, these surfaces can be viewed as having genus $g = l - [t/2]$ and $2[t/2]$ holes.

Finally, to select the admissible class of the functions Γ (defining a given model (1.1)), one is to require that in the associated large N SC series (1.9) the factor N^{2-2h} matches with the genus h of the associated worldsheet $\tilde{M}$. Then, to ensure eq. (4.7), the admissible polynomial pattern of the function

Γ should belong to the variety (with $M \in \mathbf{Z}_{\geq 1}$)

$$\Gamma(\{\tilde{b}_r\}, N, \{C_q(R)\}) = \sum_{M,\{q_k\}} \sum_{\bar{l} \geq [\frac{M}{2}]} s(\{\tilde{b}_r\}, \{q_k\}, \bar{l}) \; N^{-2\bar{l}} \prod_{k=1}^{M} \frac{C_{q_k}(R)}{N^{q_k - 1}}, \qquad \text{(E.2)}$$

where the (properly weighted by the *1/N* factors) $\bar{l} \geq 1$ terms encode the additional, compared to those encoded in (E.1), collapsed subsurfaces connecting a few sheets according to the same rules as for (E.1). In particular, the pattern (E.2)/(E.1) ensures the existence of the proper 'asymptotics' (1.6) of $\Gamma(...)$ defining the bare string tension $\sigma_0(\{\tilde{b}_r\})$ (to which only *linear* in $C_q(R)$ terms in (E.2) contribute).

Finally, with the help of some elementary identities [13] from the theory of $\chi_R(\hat{T}_{\{p\}})$ characters, the generic function (E.2) results in the generic pattern (4.6) of the generalized operator $Q_n(\Gamma)$. One can argue [14] that the required pattern (4.7) of the *N*-dependence takes place provided, in the standard $F_{\mu\nu}$-representation (1.4), the $1/N$ scaling of the coefficients g_r is constrained to yield the conventional 't Hooft pattern of the $1/N$ topological expansion.

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